Universitatea ”Ovidius” din Constanța
Institutul Studiilor Doctorale

# Teză de doctorat

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# Summary
# Contributions to the Study of Evolution Inclusions in Hilbert Spaces

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# Introduction

The study of nonlinear evolution inclusions in Hilbert spaces is a central topic in the field of partial differential equations, motivated both by its theoretical richness and its relevance to a wide range of applied problems, from mechanics and control theory to reaction-diffusion systems and wave propagation. Among the most powerful tools in this area are monotonicity methods, which have been extensively developed to handle differential inclusions governed by maximal monotone operators, possibly perturbed by nonmonotone terms.

This thesis is devoted to the study of two classes of abstract nonlinear second-order boundary value problems in real Hilbert spaces. The associated evolution inclusions are governed by maximal monotone operators and incorporate nonlinear perturbations of various types, such as nonmonotone, Lipschitz continuous, or sublinear terms.

The thesis begins with an introductory chapter comprising several key sections: *Context and Motivation*, *Related Works*, *Objectives of the Thesis*, *Structure of the Thesis* and *Scientific Contributions and Dissemination*. This is followed by a chapter on *Preliminaries*, in which essential notions and tools are grouped into three sections: *Function Spaces*, *Some Abstract Tools* and *Maximal Monotone Operators*.

The main body of the thesis is structured in two parts, each containing original contributions. *Part I: Evolution Inclusions in Hilbert Spaces with Parameters* corresponds to Chapter 1 of this summary, while *Part II: Antiperiodic Boundary Value Problems in Hilbert Spaces with Parameters* encompasses Chapters 2, 3, and 4. The results presented throughout this summary are given without proofs, though the underlying methods are usually indicated.

The thesis concludes with a final chapter that summarizes the main findings and discusses their mathematical and applicative relevance. Additionally, it outlines several open directions and perspectives for future research, including the extension of the abstract framework to Banach spaces, the study of differential inclusions with nonmonotone perturbations, and the behavior of solutions when relaxing the structural assumptions on the perturbation operators.

For conciseness and ease of presentation, this extended summary omits all section and subsection headings, but provides a comprehensive overview of the results obtained in each part of the thesis. The bibliographical resources mentioned are strictly selected based on the content presented. The complete list of publications can be found in the full version of the thesis.

We devote the rest of this chapter to introducing some of the relevant work in the literature, placing the contributions of this thesis within the context of existing studies.

We begin by precisely formulating the first class of problems considered in Part I. These involve boundary value problems with Dirichlet–Neumann conditions

$$(P_{\varepsilon\mu}) \begin{cases} -\varepsilon u''(t) + \mu u'(t) + Au(t) + Bu(t) \ni f(t),\ 0 < t < T, & (E_{\varepsilon\mu}) \\ u(0) = u_0, \quad u'(T) = 0, & (BC) \end{cases}$$

where $\varepsilon > 0$, $\mu \geq 0$ are two parameters. For $\mu > 0$, the corresponding reduced model is the Cauchy problem

$$(P_{\mu}) \begin{cases} \mu u'(t) + Au(t) + Bu(t) \ni f(t),\ 0 < t < T, & (E_{\mu}) \\ u(0) = u_0. & (IC) \end{cases}$$

We next consider the following nonlinear algebraic inclusion

$$(E_{00}) \qquad A\,u(t) + B\,u(t) \ni f(t), \quad \text{for a.e. } t \in (0, T).$$

This inclusion naturally arises as the stationary limit of the solutions to problem $(P_{\varepsilon\mu})$.

The operators $A$ and $B$ satisfy assumptions presented in Chapter 1.

We briefly review relevant literature on second-order abstract evolution inclusions. The study of inclusions of the form $u''(t) \in Au(t)$ under Dirichlet or boundedness conditions was initiated by V. Barbu [7, 6]. H. Brézis [9] extended the analysis on $[0, \infty)$ with nonlinear boundary conditions like $u'(0) \in \partial j(u(0) - a)$. R.E. Bruck [11] introduced inhomogeneous terms, considering $u''(t) \in Au(t) + f(t)$ with $a, b \in \overline{D(A)}$. More recently, similar second-order problems with Dirichlet–Neumann type conditions have been considered by L. Barbu and G. Moroşanu [3], and by G. Moroşanu and A. Petruşel [21, Lemma 4]. However, their results do not cover all the situations addressed in Chapter 1.

The convergence of solutions to problem $(P_{\varepsilon\mu})$ as $\varepsilon \to 0_+$ has been studied in several settings. M. Ahsan and G. Moroşanu [1] considered the case $\mu = 1$ with $A$ linear and strongly monotone, later extended to general maximal monotone $A$ by L. Barbu and G. Moroşanu [3]. G. Moroşanu and A. Petruşel [21] further examined the asymptotic behavior in two cases: (i) $\varepsilon \to 0$ with fixed $\mu > 0$, and (ii) fixed $\varepsilon > 0$, $\mu \to 0$. The results in Chapter 1 constitute original contributions that extend earlier findings.

Part II introduces antiperiodic boundary problems in abstract form, progressively refined across Chapters 2–4 under varying assumptions. The problem $(P_{\varepsilon\mu})_{ap}$ is formulated as follows:

$$(P_{\varepsilon\mu})_{ap} \begin{cases} -\varepsilon u'' + \mu u' + A\,u + B\,u \ni f/\text{or } F(\cdot, u(\cdot)) \ \text{ a.e. in } (0, T), \\ u(0) + u(T) = 0, \quad u'(0) + u'(T) = 0, \end{cases}$$

where $f \in L^2(0, T; H)$, $F : [0, T] \times H \to H$ is a Carathéodory mapping which verifies a sublinear growth condition, $\varepsilon > 0$, $\mu \geq 0$ are parameters. For $\mu > 0$, the corresponding reduced problem takes the form

$$(P_\mu)_{ap}\begin{cases} \mu u' + A\,u + B\,u \ni f/\text{or } F(\cdot, u(\cdot)) \ \text{ a.e. in } (0,T),\\ u(0) + u(T) = 0.\end{cases}$$

Additionally, the associated algebraic inclusion is given by

$$(E_{00})_{ap} \qquad A\,u + B\,u \ni f \quad \text{a.e. in } (0,T).$$

The (possibly set-valued) operators $A$ and $B$ are subject to distinct assumptions in each chapter.

The problem $(P_\mu)_{ap}$ with $B = 0$ was studied by Okochi [22], and by Haraux [17] in the case $Bu = \lambda u$, $\lambda > 0$. Aizicovici and Pavel [2] addressed problems $(P_{\varepsilon\mu})_{ap}$ and $(P_\mu)_{ap}$ with $B = 0$ or $B = -\partial\psi$, where $A = \partial\varphi$ dominates $B$ in a specified sense. They established existence, uniqueness, and continuous dependence for antiperiodic solutions when $B = 0$, and existence results when $B = -\partial\psi$. These works motivated further research, with notable contributions in Hilbert spaces by Chen [13], Chen et al. [12, 14, 15], and Couchouron and Precup [16]. The results obtained here complement these studies.

The behavior of solutions to $(P_{\varepsilon\mu})_{ap}$ with respect to both parameters, as well as the approximation of solutions to $(P_\mu)_{ap}$ and the algebraic inclusion $(E_{00})_{ap}$, has not been addressed for antiperiodic boundary conditions. This work fills that gap and offers new insights.

Finally, we present a central notion that sets the stage for the rest of our analysis: the concept of a strong solution. This definition provides the precise functional framework in which we formulate and study our problems in Part II.

**Definition 1** ([5], [25], [26])**.** *A function $u \in W^{2,2}(0,T;H)$ is said to be a (strong) solution to problem $(P_{\varepsilon\mu})_{ap}$ if the following conditions are all satisfied*
*(i) $u(t) \in D(A)$ for a.e. $t \in (0,T)$;*
*(ii) there exist $\xi$, $\eta \in L^2(0,T;H)$, such that*

$$\begin{gathered} -\varepsilon u''(t) + \mu u'(t) + \xi(t) - \eta(t) = f(t)(\textit{or } F(t, u(t))) \ \ \textit{and} \\ \xi(t) \in A\,u(t),\, \eta(t) \in B\,u(t) \ \ \textit{for a.e. } t \in (0,T); \end{gathered} \tag{0.1}$$

*(iii) $u(0) + u(T) = 0$, $u'(0) + u'(T) = 0$.*
*In a similar way, a function $u \in W^{1,2}(0,T;H)$ is said to be a solution to problem $(P_\mu)_{ap}$ if $u$ fulfills conditions (i), (ii) (with $\varepsilon = 0$), and $u(0) + u(T) = 0$.*

*Note that if the operators $A$ and/or $B$ are single-valued, then for all $t \in [0,T]$, we have $\xi(t) = A\,u(t)$ and/or $\eta(t) = B\,u(t)$. In this case, the definition above is simplified, with $\xi(t)$ and/or $\eta(t)$ being replaced by the expressions $A\,u(t)$ and/or $B\,u(t)$, respectively.*

**Keywords:** Evolution inclusion, Lions regularization, antiperiodic solution, maximal monotone operator, subdifferential, Lipschitz operator, semilinear parabolic equations, nonlinear ordinary differential systems, heat equation, telegraph differential system.

# Part I

Evolution Inclusions in Hilbert Spaces with Parameters

# Chapter 1

## A Class of Evolution Inclusions with Two Parameters

This chapter is dedicated to a detailed presentation of results published in a joint work with L. Barbu and G. Moroşanu in *Nonlinear Anal. Real World Appl.* [4].

**Among the original contributions in this chapter,** we mention in the summary Theorems 1.1-1.7. The rest of the obtained results are presented in the full version of the thesis.

Let $H$ denote a real Hilbert space with inner product $(\cdot,\cdot)$ and norm $\|\cdot\|$.

We consider the following boundary-value problem in the Hilbert space $H$

$$(P_{\varepsilon\mu})\ \begin{cases} -\varepsilon u''(t)+\mu u'(t)+Au(t)+Bu(t)\ni f(t),\ 0<t<T, & (E_{\varepsilon\mu})\\ u(0)=u_0,\quad u'(T)=0, & (BC)\end{cases}$$

where $T>0$ is a given time instant; $\varepsilon>0$, $\mu\geq 0$ are two parameters, and $A,B$ are operators satisfying the following hypotheses:

$(H_A)$ $A:D(A)\subset H\to H$ is a maximal monotone operator (possibly set-valued, in which case $(E_{\varepsilon\mu})$ is a differential or evolution inclusion);
$(H_B)$ $B:D(B)=H\to H$ is a Lipschitz operator, i.e., there exists a constant $L>0$ such that $\| Bx-By\|\leq L\| x-y\|$, for all $x,\ y\in H$.

Further assumptions will be imposed later in the chapter.

For $\mu>0$ we also consider the following Cauchy problem

$$(P_{\mu})\ \begin{cases} \mu u'(t)+Au(t)+Bu(t)\ni f(t),\ 0<t<T, & (E_{\mu})\\ u(0)=u_0. & (IC)\end{cases}$$

Problem $(P_\mu)$ is a reduced problem obtained by making $\varepsilon = 0$ in $(P_{\varepsilon\mu})$, which is said to be a perturbed problem associated with $(P_\mu)$. Notice that problem $(P_\mu)$ inherits from problem $(P_{\varepsilon\mu})$ only the condition $u(0) = u_0$.

We also consider the following (algebraic) inclusion

$$(E_{00}) \quad Au(t) + Bu(t) \ni f(t),\ 0 \le t \le T,$$

which is obtained by taking $\varepsilon = 0$ and $\mu = 0$ in equation $(E_{\varepsilon\mu})$.

As a first step, we introduce the definition of (strong) solutions corresponding to problems $(P_\mu)$ and $(P_{\varepsilon\mu})$.

**Definition 2** ([19, Definition 2.1, p. 47]). *Assume that assumptions $(H_A)$ and $(H_B)$ are satisfied and $u_0 \in D(A)$.*
*Let $\mu > 0$. A function $u \in W^{1,2}(0,T;H)$ is said to be a (strong) solution to problem $(P_\mu)$ if the following conditions are all satisfied*
*(i) $u(t) \in D(A)$ for a.e. $t \in (0,T)$;*
*(ii) there exists $\xi \in L^2(0,T;H)$, such that*

$$\mu u'(t) + \xi(t) + Bu(t) = f(t) \quad \text{and } \xi(t) \in A\,u(t) \text{ for a.e. } t \in (0,T); \tag{1.1}$$

*(iii) $u(0) = u_0$.*
*Let $\varepsilon > 0$ and $\mu \ge 0$. In a similar way, a function $u \in W^{2,2}(0,T;H)$ is said to be a (strong) solution to problem $(P_{\varepsilon\mu})$ if $u$ fulfills condition (i),*
*(ii)′ there exists $\xi \in L^2(0,T;H)$, such that*

$$-\varepsilon u''(t) + u'(t) + \xi(t) + Bu(t) = f(t) \text{ and } \xi(t) \in inA\,u(t) \text{ for a.e. } t \in (0,T); \tag{1.2}$$

*(iii)′ $u(0) = u_0,\ u'(T) = 0$.*

In the first section, we aim to establish existence and uniqueness results for the solutions of problem $(P_{\varepsilon\mu})$, as well as for the algebraic inclusion $(E_{00})$. We start by examining problem $(P_{\varepsilon\mu})$.

**Theorem 1.1** ([4]). *Let $\varepsilon > 0$ and $\mu \ge 0$. Assume that $(H_A)$ and $(H_B)$ are fulfilled, with the Lipschitz constant $L$ of $B$ satisfying*

$$L < \frac{2\varepsilon}{T^2}. \tag{1.3}$$

*Then, for every $u_0 \in D(A)$ and every $f \in L^2(0,T;H)$, there exists a unique solution $u = u_{\varepsilon\mu} \in W^{2,2}(0,T;H)$ to problem $(P_{\varepsilon\mu})$.*

The proof is based on using the Yosida regularization for approximating the nonlinear operator $A$ in the case $B = 0$. The existence and uniqueness results are obtained through compactness arguments, including the Arzelà–Ascoli Criterion and the demiclosedness property of maximal monotone operators. The existence and uniqueness in the case $B \neq 0$ results from a fixed-point argument, using the Banach Contraction Principle in the imposed condition on the Lipschitz constant $L$.

The condition $L < \frac{2\varepsilon}{T^2}$ plays a crucial role in the result, but it is not necessary in all cases, as the following example illustrates.

**A simple counterexample** ([4])
Consider in $H = \mathbb{R}$ the following problem

$$(P) \quad \begin{cases} -u''(t) + \mu u'(t) - u(t) = 0, \ 0 < t < T, \\ u(0) = u_0 \neq 0, \quad u'(T) = 0. \end{cases}$$

For $\mu = 0$, the general solution of the above equation satisfying $u(0) = u_0$ is given by $u(t) = u_0 \cos t + c \sin t, \ 0 \leq t \leq T$. If $T = \pi/2$, we get $u'(\pi/2) = -u_0 \neq 0$, so the problem above has no solution. Notice that condition (1.3) is not satisfied ($L = 1 > 8/\pi^2 = 2\varepsilon/T^2$).

However, it may happen that problem $(P_{\varepsilon\mu})$ has a solution, even if the condition (1.3) is not satisfied. For example, if we consider again problem $(P)$ above, with $\mu = 0$ and $T = 2\pi$, then $u = u_0 \cos t$ is the unique solution of this problem, even if condition (1.3) is not satisfied.

From the arguments implemented in the proof of Theorem 1.1, we see that if $B = 0$, the theorem holds true without condition (1.3).

Next, we continue with the equation (inclusion) $(E_{00})$. We require the following stronger assumption on $A$.
$(H_A)'$ $\quad A : D(A) \subset H \to H$ (possibly set-valued) is maximal monotone and, in addition, strongly monotone with constant $\omega > 0$; i.e.

$$(x - y, \, u - v) \geq \omega \|u - v\|^2 \text{ for all } \ u, v \in D(A) \text{ and } x \in A\,u, \, y \in A\,v. \tag{1.4}$$

Using a similar approach to the proof of Theorem 1.1, we have the following existence and uniqueness result for the solution to equation $(E_{00})$.

**Theorem 1.2** ([4])**.** *Assume that $(H_A)'$ and $(H_B)$ are both fulfilled, with constants $\omega > 0$ and $L > 0$ satisfying $L < \omega$. Then, for every $f \in W^{1,p}(0, T; H)$ and every $p \in (1, \infty)$, equation $(E_{00})$ has a unique solution $u \in W^{1,p}(0, T; H)$.*

After establishing the well-posedness of problem $(P_{\varepsilon\mu})$, we proceed to analyze the dependence of the solution on the parameters $\varepsilon > 0$ and $\mu \geq 0$. More precisely, we show that the solution $u_{\varepsilon\mu}$ varies continuously with respect to $\varepsilon$ and $\mu$, as they approach fixed reference values $\varepsilon_0 > 0$ and $\mu_0 \geq 0$. This result is obtained by deriving suitable estimates in the Hilbert space $L^2(0, T; H)$ and careful use of interpolation arguments.

Having established the continuity of the solution $u_{\varepsilon\mu}$ with respect to the parameters $\varepsilon$ and $\mu$, we now investigate the asymptotic behavior of this solution as $\varepsilon \to 0_+$ and $\mu \to \mu_0 > 0$. This analysis aims to rigorously justify the convergence from the second-order regularized problem $(P_{\varepsilon\mu})$ to the first-order problem $(P_\mu)$ as the regularization vanishes.

**Theorem 1.3** ([4]). *Assume that $(H_A)$ and $(H_B)$ hold, $\mu_0$ is a fixed positive constant, $u_0 \in D(A)$, and $f \in W^{1,1}(0,T;H)$. Then, for every $\varepsilon > 0$ small enough and $\mu > 0$ close to $\mu_0$, the problems $(P_{\varepsilon\mu})$ and $(P_\mu)$ have unique solutions, $u_{\varepsilon\mu} \in W^{2,2}(0,T;H)$ and $u_\mu \in W^{1,\infty}(0,T;H)$, respectively, and the following estimate holds:*

$$\|u_{\varepsilon\mu} - u_{\mu_0}\|_{C([0,T];H)} = \mathcal{O}(\varepsilon^{1/4}) + \mathcal{O}(|\ \mu - \mu_0\ |^{1/2}). \tag{1.5}$$

This convergence result is obtained by decomposing the analysis into two stages. The first part proves that the solution to problem $(P_\mu)$ converges continuously to the solution of $(P_{\mu_0})$ as $\mu \to \mu_0$, using a Bielecki norm. The second stage estimates the difference between $u_{\varepsilon\mu}$ and $u_\mu$ by applying an exponentially weighted transformation, followed by estimates deduced from the monotony property of operator $A$ and the Lipschitz property of operator $B$. These two estimates are then combined to conclude the result.

**Remark 1.1** ([4]). *If $\varepsilon > 0$ is a small parameter and $\mu$ is a fixed positive number (in particuar $\mu = 1$), then the (perturbed) problem $(P_{\varepsilon\mu}) \equiv (E_{\varepsilon\mu}) + (BC)$ is a Lions type regularization of the (reduced) problem $(P_\mu) \equiv (E_\mu) + (IC)$ above. For the case $A$ linear and $B = 0$, see [18].*

**Remark 1.2.** *If $A$ is a subdifferential operator, a similar convergence holds, as stated below.*

**Theorem 1.4** ([4]). *Assume that $A$ is the subdifferential of a proper, convex, and lower semicontinuous function $\varphi : H \to (-\infty, +\infty]$, and $(H_B)$ holds. Let $\mu_0 > 0$ be fixed, and let $u_0 \in D(A)$ and $f \in L^2(0,T;H)$. Then, for every $\varepsilon > 0$ small enough and $\mu > 0$ close to $\mu_0$, the problems $(P_{\varepsilon\mu})$ and $(P_\mu)$ admit unique solutions $u_{\varepsilon\mu} \in W^{2,2}(0,T;H)$ and $u_\mu \in W^{1,2}(0,T;H)$, respectively, and the estimate (1.5) holds.*

This version of the convergence result relies on analogous techniques, adapted to the subdifferential structure of $A$. In particular, the norm estimate for the derivative of $u_\mu$ is obtained using a well-known result (see [10, Theorem 3.6, p. 72]), in combination with the general continuity argument from the first stage of the proof of Theorem 1.3.

The following section concerns the asymptotic behavior of the solution $u_{\varepsilon\mu}$ to problem $(P_{\varepsilon\mu})$ as the parameters vanish, i.e., as $\varepsilon \to 0_+$ and $\mu \to 0_+$. In this fashion, the dynamics are expected to converge to those of the algebraic inclusion $(E_{00})$. However, since $u_{\varepsilon\mu}(0) = u_0$ while $u$, the solution of $(E_{00})$, typically does not satisfy $u(0) = u_0$, a boundary layer develops near $t = 0$. To compensate for this mismatch, a corrector is introduced.

We first consider a simplified model where the operator $A$ is linear and scalar, i.e., $A = \omega I$ with $\omega > 0$, and $B = 0$. In this case, the reduced equation $(E_{00})$ becomes $\omega u = f$, and the exact corrector can be explicitly constructed. Consider the problem

$$\begin{cases} -\varepsilon u''_{\varepsilon\mu} + \mu u'_{\varepsilon\mu} + \omega u_{\varepsilon\mu} = f,\ t \in (0,T), \\ u_{\varepsilon\mu}(0) = u_0, \quad u'_{\varepsilon\mu}(T) = 0. \end{cases} \tag{1.6}$$

In this case, we proved the following convergence result.

**Theorem 1.5** ([4])**.** *Let $\varepsilon > 0$ and $\mu > 0$. Then for every $u_0 \in H$ and every $f \in W^{1,2}(0,T;H)$, problem (1.6) and equation $(E_{00})$ have unique solutions $u_{\varepsilon\mu} \in W^{2,2}(0,T;H)$ and respectively $u = (1/\omega)f \in W^{1,2}(0,T;H)$. Moreover, for $\varepsilon,\ \mu \ll 1$, the following estimate holds*

$$\|u_{\varepsilon\mu} - (u+\alpha)\|_{C([0,T];H)} = \mathcal{O}\left(\mu^{1/2} + \varepsilon^{1/4} + \frac{\mu}{\varepsilon^{1/4}} + \frac{\varepsilon^j}{\mu^{1/2}}\right) \ \forall j \geq 1. \tag{1.7}$$

The proof is based on constructing a corrector function $\alpha(t)$ that absorbs the boundary layer at $t = 0$ and satisfies an ordinary differential equation in $H$.

We now move to a more general setting, in which $A$ is a linear and maximal $\omega-$strongly monotone operator and $B$ is Lipschitz. Under these assumptions, the convergence still holds, though in the weaker norm of $L^2(0,T;H)$:

**Theorem 1.6** ([4])**.** *Let $0 < \varepsilon,\ \mu \ll 1$, such that $\varepsilon < \mu^2/(4L)$. Assume that $(H_A)'$ and $(H_B)$ are fulfilled and, in addition, $L < \omega$. Then, for every $u_0 \in D(A)$ and every $f \in W^{1,2}(0,T;H)$, problem $(P_{\varepsilon\mu})$ and equation $(E_{00})$ have unique solutions $u_{\varepsilon\mu} \in W^{2,2}(0,T;H)$ and respectively $u \in W^{1,2}(0,T;H)$. Moreover, if in addition $A$ is a linear operator, the following estimate holds*

$$\| u_{\varepsilon\mu} - u \|_{L^2(0,T;H)} = \mathcal{O}(\mu^{1/2}). \tag{1.8}$$

Notice that, if $(H_B)$ is replaced by
$(H_B)'$ $\quad B : H \to H$ is monotone and Lipschitz on bounded sets,
we have the following approximation result.

**Theorem 1.7** ([4])**.** *Let $0 < \varepsilon,\ \mu \ll 1$. Assume that $(H_A)'$ and $(H_B)'$ are fulfilled. Then, for every $u_0 \in D(A)$ and every $f \in W^{1,2}(0,T;H)$, problem $(P_{\varepsilon\mu})$ and equation $(E_{00})$ have unique solutions $u_{\varepsilon\mu} \in W^{2,2}(0,T;H)$ and respectively $u \in W^{1,2}(0,T;H)$. Moreover, if $A$ is a linear operator, the following estimate holds*

$$\| u_{\varepsilon\mu} - u \|_{L^2(0,T;H)} = \mathcal{O}\left(\mu^{1/2} + \varepsilon^{1/4} + \frac{\varepsilon^j}{\mu^{1/2}}\right) \ \forall j \geq 1. \tag{1.9}$$

In this setting, the proof of Theorem 1.6 can be repeated almost entirely, with the key modification arising in the control of the nonlinear term involving $B$.

**Remark 1.3** ([4])**.** *Unfortunately, the estimates provided by Theorems 1.6 and 1.7 are in $L^2(0,T;H)$, not in $C([0,T];H)$. Obtaining estimates in the $C([0,T];H)$-norm remains an open problem.*

In the final section of this chapter, we apply our abstract findings to two applications, the regularization of the nonlinear heat equation and the regularization of the telegraph system.

# Part II

Antiperiodic Boundary Value Problems in Hilbert Spaces with Parameters

# Chapter 2

## On One-Parameter Evolution Inclusions with Antiperiodic Conditions

The results presented in this chapter are part of a manuscript currently under review in *Monatshefte für Mathematik* [26].

**Among the original contributions in this chapter**, we mention Theorems 2.1-2.4. The rest of the obtained results are presented in the full version of the thesis.

Recall that $H$ denotes a real Hilbert space with inner product $(\cdot,\cdot)$ and norm $\|\cdot\|$.

We consider the following antiperiodic boundary value problems in $H$

$$(P_\varepsilon)_{ap} \quad \begin{cases} -\varepsilon u''(t) + u'(t) + Au(t) + Bu(t) \ni F(t,u(t)) \quad \text{for a.e. } t \in (0,T), & (E_\varepsilon) \\ u(0)+u(T)=0, \quad u'(0)+u'(T)=0, \end{cases}$$

and

$$(P_0)_{ap} \quad \begin{cases} u'(t) + Au(t) + Bu(t) \ni F(t,u(t)) \quad \text{for a.e. } t \in (0,T), & (E_0) \\ u(0)+u(T)=0. \end{cases}$$

To study problems $(P_\varepsilon)_{\text{ap}}$ and $(P_0)_{\text{ap}}$, we place ourselves in the following abstract framework

$(H_F)$ The mapping $F:[0,T]\times H \to H$ is a Carathéodory function and satisfies the following sublinear growth condition

$$\| F(t,v) \| \leq L \| v \| + l(t) \text{ for a.e. } t \in (0,T) \text{ and for all } v \in H, \tag{2.1}$$

where $L \geq 0$ and $l \in L^2(0,T)$, with $l(t) \geq 0$ for a.e. $t \in (0,T)$.

$(H_A)$ The operator $A = \partial\varphi$, where $\varphi : H \to [0,+\infty]$ is an even, proper, convex, and lower semicontinuous function, such that $\varphi(0) = 0$. In addition, $\varphi$ satisfies the following condition

$$(H_{\varphi_c}) \text{ for every } r > 0, \text{ the set } \{x \in D(\varphi); \parallel x \parallel + \varphi(x) \leq r\} \text{ is compact in } H.$$

The operator $B$ is assumed to satisfy one of the following two conditions:

$(H_B)_1$ $B = -\partial\psi$, where $\psi : H \to \mathbb{R}$ is an even, continuously differentiable function, and for all $r > 0$, there exists $K_r > 0$ such that $\parallel \partial\psi(v) \parallel \leq K_r$ for all $v \in H$ with $\parallel v \parallel \leq r$;

$(H_B)_2$ $B = -\partial\psi$, where $\psi : H \to (-\infty, +\infty]$ is an even, proper, convex, and lower semicontinuous function satisfying

$(H_\psi)$ $D(\partial\varphi) \subset D(\partial\psi)$, and for all $r > 0$, there exist a constant $\rho_r \in [0, 1)$, and a non-decreasing function $\gamma : [0, \infty) \to [0, \infty)$, such that for all $u \in D(\partial\varphi)$ with $\parallel u \parallel \leq r$, the following condition is satisfied

$$\parallel (\partial\psi)^0(u) \parallel^2 \leq \rho_r \parallel (\partial\varphi)^0(u) \parallel^2 + \gamma(\parallel u \parallel)\big(\varphi(u) + 1\big) \tag{2.2}$$

(here $(\partial\varphi)^0$ stands for the minimal section of $\partial\varphi$ and similarly for $\psi$).

In the first section, we address existence and uniqueness results concerning problems $(P_\varepsilon)_{ap}$ and $(P_0)_{ap}$, along with uniform estimates that are essential for analyzing the limiting behavior as $\varepsilon \to 0$. We begin by working under the assumption that the operator $B$ satisfies $(H_B)_1$ and establish the first existence result within this framework.

**Theorem 2.1** ([26]). *Assume that $(H_A)$, $(H_B)_1$, and $(H_F)$ hold, with the constant $L$ in $(H_F)$ satisfying $L < \pi/T$. Then, for every $\varepsilon > 0$, problems $(P_\varepsilon)_{ap}$ and $(P_0)_{ap}$ have at least one solution $u_\varepsilon \in W^{2,2}(0, T; H)$ and $u \in W^{1,2}(0, T; H)$, respectively. Furthermore, these solutions satisfy the following estimates*

$$\begin{aligned}&\parallel u_\varepsilon' \parallel_{L^2(0,T;H)} \leq \parallel l \parallel_{L^2(0,T)} /k_L, \ \parallel u_\varepsilon \parallel_{C([0,T];H)} \leq \mathcal{R}_0,\\ &\parallel \partial\psi(u_\varepsilon) \parallel_{L^2(0,T;H)} \leq C_1, \ \varepsilon^2 \parallel u_\varepsilon'' \parallel^2_{L^2(0,T;H)} + \parallel \xi_\varepsilon \parallel^2_{L^2(0,T;H)} \leq C_2^2, \ \parallel \varphi(u_\varepsilon) \parallel_{C[0,T]} \leq C_3,\end{aligned} \tag{2.3}$$

*with $u_\varepsilon$, $\xi_\varepsilon$, $\varepsilon$ replaced by $u$, $\xi$, $0$, respectively, if $u$ is a solution to $(P_0)_{ap}$,*

*where $k_L = 1 - LT/\pi$, $\mathcal{R}_0 \overset{not.}{=} \sqrt{T} \parallel l \parallel_{L^2(0,T)} /(2k_L)$, and $C_i$, $i = \overline{1,3}$ are positive constants depending on $T$, $L$, $\|l\|_{L^2(0,T)}$, $\psi(0)$, and the constant from assumption $(H_B)_1$ with $r = \mathcal{R}_0$, but are independent of $\varepsilon$.*
*In (2.3), $\xi_\varepsilon$ and $\xi$ denote the sections of $\partial\varphi(u_\varepsilon)$ and $\partial\varphi(u)$, respectively (see Definition 1).*

The existence of solutions for problems $(P_\varepsilon)_{ap}$ and $(P_0)_{ap}$ is established using a compactness argument based on the Schaefer Fixed Point Theorem. Key estimates are derived via Poincaré-type inequalities for $H-$valued antiperiodic functions, monotonicity techniques, and regularity arguments involving the subdifferentials $\partial\varphi$ and $\partial\psi$.

A similar result to Theorem 2.1 holds, assuming $B$ satisfies $(H_B)_2$ instead of $(H_B)_1$.

**Theorem 2.2** ([26]). *Assume that $(H_A)$, $(H_B)_2$, and $(H_F)$ hold, with the constant $L$ in $(H_F)$ satisfying $L < \pi/T$. Then, for every $\varepsilon > 0$, the problems $(P_\varepsilon)_{ap}$ and $(P_0)_{ap}$ have at least one solution $u_\varepsilon \in W^{2,2}(0,T;H)$ and $u \in W^{1,2}(0,T;H)$, respectively. Furthermore, these solutions satisfy estimates $(2.3)_{1,2}$, and additionally*

$$\| \xi_\varepsilon \|_{L^2(0,T;H)} \leq C_1, \ \| \eta_\varepsilon \|_{L^2(0,T;H)} \leq C_2, \ \| \varphi(u_\varepsilon) \|_{C[0,T]} \leq C_3, \ \varepsilon \| u''_\varepsilon \|_{L^2(0,T;H)} \leq C_4, \tag{2.4}$$

$$\textit{with } u_\varepsilon,\, \xi_\varepsilon,\, \eta_\varepsilon,\, \varepsilon \textit{ replaced by } u,\, \xi,\, \eta,\, 0, \textit{ respectively, if } u \textit{ is a solution to } (P_0)_{ap}.$$

*Here $C_i$, $i = \overline{1,4}$ are positive constants depending on $T$, $L$, $\|l\|_{L^2(0,T)}$, and the constant and function from assumption $(H_\psi)$ with $r = \mathcal{R}_0$ (previously defined in Theorem 2.1), but are independent of $\varepsilon$.*
*In (2.4), $\xi_\varepsilon$ and $\eta_\varepsilon$ denote the sections of $\partial\varphi(u_\varepsilon)$ and $\partial\psi(u_\varepsilon)$, while $\xi$ and $\eta$ correspond to the sections of $\partial\varphi(u)$ and $\partial\psi(u)$, respectively.*

The proof of this result is based on using the Moreau-Yosida regularisation for function $\psi$, which led to a family of approximate problems. Uniform estimates are derived using Poincaré-type inequalities for $H-$valued antiperiodic functions, properties of subdifferentials and structural assumptions like $(H_\psi)$. Using compactness methods and the demiclosedness property of maximal monotone operators, we obtain, passing to the limit, the existence of solutions to the original problem, along with the corresponding estimates. This technique applies similarly for both the perturbed and unperturbed problems.

**Remark 2.1** ([26]). *An operator satisfying $(H_B)_1$ is not a particular case of one satisfying $(H_B)_2$. For example, consider $H = L^2(\Omega)$, where $\Omega$ is a nonempty bounded domain in $\mathbb{R}^N$, and let $q \in (1,2]$. Take $a \in L^\infty(\Omega)$, such that $m(\{x \in \Omega;\, a(x) > 0\}) > 0$ and $m(\{x \in \Omega;\, a(x) < 0\}) > 0$. Define the even, continuously differentiable function $k : H \to \mathbb{R}$, $k(u) = q^{-1}\int_\Omega a(x) \mid u \mid^q dx$, and set $B = -\partial k : H \to H$. Then, for all $u \in H$, we have $B\,u = -a(x) \mid u \mid^{q-1} sgn\, u$. By Hölder's inequality, $B$ satisfies $(H_B)_1$, but since $k$ is neither convex nor concave, $B$ does not satisfy $(H_B)_2$.*

**Remark 2.2** ([26]). *If $F : \mathbb{R} \times H \to H$ satisfies assumption $(H_F)$ on $[0,T]$ and, in addition,*

$$F(t+T,u) + F(t,-u) = 0 \quad \textit{for a.e. } t \in \mathbb{R} \textit{ and } u \in H,$$

*and if $B$ is an odd operator, then the solutions obtained in the results of this section can be extended to all of $\mathbb{R}$ by $T-$antiperiodicity.*

In analyzing the behavior of the solutions to $(P_\varepsilon)_{ap}$ with respect to $\varepsilon$, it is worth emphasizing that under the assumptions of Theorem 2.1 and Theorem 2.2, the problems $(P_\varepsilon)_{ap}$ and $(P_0)_{ap}$ may exhibit genuine nonuniqueness of solutions, as discussed in references [17] and [24]. Nonetheless, we have the following result concerning the convergence of sequences of solutions to problem $(P_\varepsilon)_{ap}$.

**Theorem 2.3** ([26]). *Let $\varepsilon_0 \geq 0$ be fixed. Assume that $(H_A)$ holds, along with either $(H_B)_1$ or $(H_B)_2$. In addition, $(H_F)$ is fulfilled, with the constant $L$ satisfying $L < \pi/T$. For every $\varepsilon > 0$, let $u_\varepsilon$ be a solution to problem $(P_\varepsilon)_{ap}$, given by Theorem 2.1 under $(H_B)_1$, or by Theorem 2.2 under $(H_B)_2$. Then, for every sequence $0 < \varepsilon_n \to \varepsilon_0$, there exists a subsequence (not relabeled), such that*

$$\begin{aligned} &u_{\varepsilon_n} \to u \text{ in } C([0,T];H),\ u'_{\varepsilon_n} \to u' \text{ weakly in } W^{1,2}(0,T;H) \quad \text{if } \varepsilon_0 > 0, \\ &u_{\varepsilon_n} \to u \text{ in } C([0,T];H),\ u'_{\varepsilon_n} \to u' \text{ weakly in } L^2(0,T;H) \quad \text{if } \varepsilon_0 = 0, \end{aligned} \tag{2.5}$$

*where the limit $u$ is a solution to $(P_{\varepsilon_0})_{ap}$ if $\varepsilon_0 > 0$, and a solution to $(P_0)_{ap}$ if $\varepsilon_0 = 0$.*

The proof relies on the uniform estimates obtained in Theorems 2.1 and 2.2, compactness arguments and the demiclosedness property of maximal monotone operators.

We also provide two sets of sufficient conditions to guarantee the uniqueness of solutions to problems $(P_\varepsilon)_{ap}$ and $(P_0)_{ap}$. These conditions allow us to establish that the solution to $(P_\varepsilon)_{ap}$ is continuous with respect to $\varepsilon$ and approximates the solution to $(P_0)_{ap}$ as $\varepsilon \to 0_+$.

We impose the following stronger condition on $F$, instead of $(H_F)$.

$(H_F)'$ $F$ is a Carathéodory mapping and instead of 2.1, we assume that $F$ satisfies

$$\| F(t,v) - F(t,w) \| \leq \bar{L} \| v - w \|, \tag{2.6}$$

for a.e. $t \in (0,T)$ and all $v, w \in H$, where $\bar{L}$ is a positive constant.

We denote $\mathcal{R}_0 = \sqrt{T} \| F(t,0) \|_{L^2(0,T)} /(2k_{\bar{L}})$, where $k_{\bar{L}} = 1 - \bar{L}T/\pi$, as defined in Theorem 2.1.

**Theorem 2.4** ([26]). *Let $\varepsilon_0 \geq 0$ be fixed. Assume that $(H_A)$ and $(H_F)'$ hold, along with either $(H_B)_1$ or $(H_B)_2$. Furthermore, suppose that one of the following conditions is fulfilled:*
*$(h_1)$ $A$ is a strongly monotone operator with constant $\omega > 0$, $B$ is Lipschitz continuous on the ball $\overline{B}_H(0,\mathcal{R}_0)$ with the Lipschitz constant $L_B$, and $\bar{L} + L_B < \omega$;*
*$(h_2)$ $A$ is a single-valued linear operator, $B$ is Lipschitz continuous on the ball $\overline{B}_H(0,\mathcal{R}_0)$ with the Lipschitz constant $L_B$, and $(\bar{L} + L_B)T < \pi$.*
*Then, for every $\varepsilon > 0$, the solutions $u_\varepsilon$ and $u$ to problems $(P_\varepsilon)_{ap}$ and $(P_0)_{ap}$, respectively (as established by Theorem 2.1 under $(H_B)_1$, or by Theorem 2.2 under $(H_B)_2$), are unique. In addition, the following estimates and approximations are valid*

$$\begin{aligned} &\| u_\varepsilon - u_{\varepsilon_0} \|_{C([0,T];H)} \leq \frac{\sqrt{T}}{2} \| u'_\varepsilon - u'_{\varepsilon_0} \|_{L^2(0,T;H)} = \mathcal{O}(| \varepsilon - \varepsilon_0 |) \quad \text{and} \\ &u_\varepsilon \to u_{\varepsilon_0} \ \text{ in } \ C^1([0,T];H) \quad \text{as} \quad \varepsilon \to \varepsilon_0 > 0, \\ &\| u_\varepsilon - u \|_{L^2(0,T;H)} = \mathcal{O}(\sqrt{\varepsilon}) \quad \text{if } (h_1) \text{ holds and } \ \varepsilon \to 0_+, \\ &u_\varepsilon \to u \quad \text{in} \quad C([0,T];H) \quad \text{as} \quad \varepsilon \to 0_+. \end{aligned} \tag{2.7}$$

The proof relies on Poincaré-type inequalities for $H-$valued antiperiodic functions and monotonicity techniques to establish uniqueness of solutions. Separate estimates are derived under structural conditions $(h_1)$ and $(h_2)$, depending on whether the operator $A$ is strongly monotone or linear. Regularity and compactness arguments, together with the Arzelà–Ascoli Criterion, are employed to prove convergence in $C^1([0,T];H)$ as $\varepsilon \to \varepsilon_0 > 0$, and convergence in $C([0,T];H)$ as $\varepsilon \to 0_+$.

The final section is devoted to applications of the abstract results to concrete problems. Specifically, we examine the semilinear heat equation under time-antiperiodic boundary conditions, as well as antiperiodic systems of ordinary differential equations.

# Chapter 3

## On Two-Parameter Evolution Inclusions with Antiperiodic Conditions

Based on a joint work with L. Barbu and G. Moroşanu, accepted for publication in *Communications in Contemporary Mathematics* [5], this chapter investigates a second-order antiperiodic boundary value problem that focuses on the case where both parameters $\varepsilon > 0$ and $\mu \geq 0$ are present. In addition, we examine the associated reduced problem obtained by setting $\varepsilon = 0$, as well as the limiting algebraic inclusion corresponding to the case $\varepsilon = \mu = 0$.

**Among the original contributions from this chapter**, we mention in the summary Theorems 3.1-3.9. The rest of the results obtained can be consulted in the integral content of the thesis.

Recall that $H$ denotes a real Hilbert space with inner product $(\cdot,\cdot)$ and norm $\|\cdot\|$.

Let $T > 0$ be a given final time. We first consider the following second-order antiperiodic problem in a real Hilbert space $H$:

$$(P_{\varepsilon\mu})_{ap} \quad \begin{cases} -\varepsilon u''(t) + \mu u'(t) + Au(t) + Bu(t) \ni f(t) \quad \text{for a.e. } t \in (0,T), & (E_{\varepsilon\mu}) \\ u(0) + u(T) = 0, \quad u'(0) + u'(T) = 0, \end{cases}$$

where $\varepsilon > 0$, $\mu \geq 0$, and $f \in L^2(0,T;H)$.

In the case $\mu > 0$, we also consider the associated first-order antiperiodic problem:

$$(P_{\mu})_{ap} \quad \begin{cases} \mu u'(t) + A\,u(t) + B\,u(t) \ni f(t) \quad \text{for a.e. } t \in (0,T), & (E_{\mu}) \\ u(0) + u(T) = 0. \end{cases}$$

In addition, we analyze the limiting case $\varepsilon = \mu = 0$, which leads to the following algebraic inclusion:

$$(E_{00})_{ap} \qquad A\,u(t) + B\,u(t) \ni f(t) \text{ for a.e. } t \in (0,T).$$

The general framework in which we analyze these problems is given by the following assumptions:
$(H_{\varepsilon\mu})$ the parameters $\varepsilon$ and $\mu$ satisfy $\varepsilon \geq 0$, $\mu \geq 0$, and $\varepsilon + \mu > 0$;
$(H_A)$ $A = \partial\varphi$, where $\varphi : H \to [0, +\infty]$ is a proper, convex, lower semicontinuous, and even function, such that $\varphi(0) = 0$;
$(H_B)$ $B : H \to H$ is a continuous operator satisfying the sublinear growth condition

$$\| B\,u \| \leq L \| u \| + l \text{ for all } u \in H,$$

for some constants $L > 0$ and $l \geq 0$.

$$(H_{\varphi_c}) \text{ for every } \gamma > 0, \text{the set } \{x \in D(\varphi);\, \| x \| \leq \gamma,\ \varphi(x) \leq \gamma\} \text{ is compact in } H.$$

Additional assumptions on the operators $A$ and $B$ will be introduced later.

The first section is dedicated to proving existence and finding uniform estimates of solutions to problems $(P_{\varepsilon\mu})_{ap}$ and $(P_\mu)_{ap}$. These estimates will later be used to analyze the behavior of strong solutions to $(P_{\varepsilon\mu})_{ap}$ with respect to $\varepsilon$ and $\mu$.

We begin by establishing the existence of at least one strong solution to problem $(P_{\varepsilon\mu})_{ap}$, along with uniform estimates with respect to the parameter $\mu$.

**Theorem 3.1** ([5])**.** *Let $\varepsilon > 0$ be fixed. Assume that $(H_A)$, $(H_{\varphi_c})$, and $(H_B)$ are fulfilled, with the constant $L$ of $B$ satisfying*

$$L < \frac{\pi^2 \varepsilon}{T^2}. \tag{3.1}$$

*Then for every $\mu \geq 0$ and $f \in L^2(0,T;H)$, the problem $(P_{\varepsilon\mu})_{ap}$ has at least one strong solution $u_{\varepsilon\mu} \in W^{2,2}(0,T;H)$, such that $\xi_{\varepsilon\mu} \in L^2(0,T;H)$, where $\xi_{\varepsilon\mu}(t)$ denotes the section of $\partial\varphi(u_{\varepsilon\mu}(t))$ as in (1.1) for a.e. $t \in (0,T)$. In addition, for every $\mu \geq 0$, the following estimates hold*

$$\| u''_{\varepsilon\mu} \|_{L^2(0,T;H)} \leq C_{1\varepsilon},\ \| u'_{\varepsilon\mu} \|_{L^2(0,T;H)} \leq C_{2\varepsilon},\ \ \| u_{\varepsilon\mu} \|_{C([0,T];H)} \leq C_{3\varepsilon}, \tag{3.2}$$

$$\| \xi_{\varepsilon\mu} \|_{L^2(0,T;H)} \leq C_{4\varepsilon},\ \| \varphi(u_{\varepsilon\mu}) \|_{L^\infty(0,T)} \leq C_{5\varepsilon}, \tag{3.3}$$

*where $C_{i\varepsilon}$, $i = \overline{1,5}$, are positive constants depending on $\varepsilon$, $T$, $\| f \|_{L^2(0,T;H)}$, $L$, and $l$, but independent of $\mu$.*

The proof of Theorem 3.1 relies on a fixed point approach, specifically the Schaefer Fixed Point Theorem, to establish the existence of a strong solution. Key tools include monotonicity arguments, Poincaré-type inequalities, and embedding results for Sobolev spaces.

**Remark 3.1** ([5])**.** *Theorem 3.1 still holds if we drop the assumption $(H_{\varphi_c})$ but instead impose a stronger condition on $B$; namely, the assumption $(H_B)$ is replaced by*

$(H_B)'$ *$B : H \to H$ is a Lipschitz continuous operator, with Lipschitz constant $\overline{L}$.*

**Remark 3.2** ([5]). *Consider the problem $(P_{\varepsilon\mu})_{ap}$ with $A = 0$, $\mu = 0$, and $B\,u = \overline{L}u$, where $\overline{L} < \varepsilon\pi^2/T^2$. So we are in the framework of Remark 3.1 above. It is well known that $\varepsilon\pi^2/T^2$ is the smallest eigenvalue of the problem*

$$\varepsilon u''(t) = \lambda u(t),\ t \in (0,T),\ u(0) + u(T) = 0,\ u'(0) + u'(T) = 0.$$

*Hence, by virtue of the Fredholm Alternative (see, e.g., [20, Theorem 7.10]), the problem*

$$-\varepsilon u''(t) - \varepsilon\frac{\pi^2}{T^2}u(t) = f(t),\ t \in (0,T),\ u(0) + u(T) = 0,\ u'(0) + u'(T) = 0$$

*is not solvable for all $f$.*

*So, the condition $\overline{L} < \varepsilon\pi^2/T^2$ is suitable if $B = \overline{L}u$ and of course the same condition is suitable in the framework of Remark 3.1.*

The next result involves a sufficient condition for the existence of solutions to problems $(P_{\varepsilon\mu})_{ap}$ and $(P_\mu)_{ap}$ when the constant $L$ is assumed to be "sufficiently small" with respect to $\mu$. In addition, uniform estimates with respect to $\varepsilon$ are obtained and will be employed in the following sections.

**Theorem 3.2** ([5]). *Let $\mu > 0$ be fixed. Assume that $(H_A)$, $(H_{\varphi_c})$, and $(H_B)$ are fulfilled, with the constant $L$ of $B$ satisfying*

$$L < \frac{\pi\mu}{T}. \tag{3.4}$$

*Then for every $\varepsilon > 0$ and $f \in L^2(0,T;H)$, the problems $(P_{\varepsilon\mu})_{ap}$ and $(P_\mu)_{ap}$ have at least one strong solution $u_{\varepsilon\mu} \in W^{2,2}(0,T;H)$ and $u_\mu \in W^{1,2}(0,T;H)$, respectively, such that $\xi_{\varepsilon\mu}$, $\xi_\mu \in L^2(0,T;H)$, where $\xi_{\varepsilon\mu}(t)$ and $\xi_\mu(t)$ denote the sections of $\partial\varphi(u_{\varepsilon\mu}(t))$ and $\partial\varphi(u_\mu(t))$, respectively, as in (1.1) for a.e. $t \in (0,T)$. In addition, for every $\varepsilon \geq 0$, the following estimates hold*

$$\| u'_{\varepsilon\mu} \|_{L^2(0,T;H)} \leq C_{1\mu},\ \| u_{\varepsilon\mu} \|_{C([0,T];H)} \leq C_{2\mu},\ \| \xi_{\varepsilon\mu} \|_{L^2(0,T;H)} \leq C_{3\mu}, \tag{3.5}$$

$$\varepsilon \| u''_{\varepsilon\mu} \|_{L^2(0,T;H)} \leq C_{4\mu},\ \| \varphi(u_{\varepsilon\mu}) \|_{L^\infty(0,T)} \leq C_{5\mu}, \tag{3.6}$$

$$\text{with } u_{\varepsilon\mu} \text{ replaced by } u_\mu \text{ if } \varepsilon = 0 \text{ (except } (3.6)_1),$$

*where $C_{i\mu}$, $i = \overline{1,5}$, are positive constants depending on $\mu$, $T$, $\| f \|_{L^2(0,T;H)}$, $L$, and $l$, but independent of $\varepsilon$.*

The proof of Theorem 3.2 builds upon the techniques used in Theorem 3.1, employing a fixed point argument to ensure existence, followed by Poincaré-type inequalities for $H-$antiperiodic functions to derive uniform bounds.

**Remark 3.3** ([5]). *Notice that Theorem 3.2 remains valid if we remove $(H_{\varphi_c})$ and instead introduce stronger assumptions on $A$ and $B$. Specifically, we assume that $(H_A)$ is satisfied, along with the additional condition that the operator $A$ is linear. Furthermore, we require that $(H_B)'$ holds with a Lipschitz constant of $B$, $\overline{L}$ satisfying $\overline{L} < \pi\mu/T$.*

The conclusions of Theorems 3.1 and 3.2 are still valid under some alternative assumptions. Specifically, we assume that $(H_A)$ is replaced by the stronger assumption
$(H_A)'$ Assumption $(H_A)$ holds and, in addition, $A$ is strongly monotone with constant $\omega > 0$ (see condition (1.4)).

In this case, one can prove a result similar to the previous ones, in which the constant $L$ satisfies a condition which involves only $\omega$.

**Theorem 3.3** ([5]). *Assume that $(H_A)'$, $(H_{\varphi_c})$, and $(H_B)$ are fulfilled, with constants $L$ and $\omega$ satisfying*

$$L < \omega. \tag{3.7}$$

*Then for every $\varepsilon$ and $\mu$ satisfying $(H_{\varepsilon\mu})$ and $f \in L^2(0,T;H)$, the problems $(P_{\varepsilon\mu})_{ap}$ and $(P_\mu)_{ap}$ have at least one strong solution $u_{\varepsilon\mu} \in W^{2,2}(0,T;H)$ and $u_\mu \in W^{1,2}(0,T;H)$, respectively, such that $\xi_{\varepsilon\mu}$, $\xi_\mu \in L^2(0,T;H)$, where $\xi_{\varepsilon\mu}(t)$ and $\xi_\mu(t)$ denote the sections of $\partial\varphi(u_{\varepsilon\mu}(t))$ and $\partial\varphi(u_\mu(t))$, respectively, as in (1.1) for a.e. $t \in (0,T)$. In addition, the following estimates hold*

$$\| u_{\varepsilon\mu} \|_{L^2(0,T;H)} \le C_1,\ \varepsilon^2 \| u''_{\varepsilon\mu} \|^2_{L^2(0,T;H)} + \mu^2 \| u'_{\varepsilon\mu} \|^2_{L^2(0,T;H)} + \| \xi_{\varepsilon\mu} \|^2_{L^2(0,T;H)} \le C_2^2, \tag{3.8}$$

*for all $\varepsilon$ and $\mu$ satisfying $(H_{\varepsilon\mu})$, with $u_{\varepsilon\mu}$ replaced by $u_\mu$ if $\varepsilon = 0$;*

$$\| u_{\varepsilon\mu} \|_{C([0,T];H)} \le C_2\sqrt{T}/(2\mu),\ \| \varphi(u_{\varepsilon\mu}) \|_{L^\infty(0,T)} \le C\, C_2\left(C_2/\mu + C_1\right), \tag{3.9}$$

*for all $\varepsilon \ge 0$ and $\mu > 0$, with $u_{\varepsilon\mu}$ replaced by $u_\mu$ if $\varepsilon = 0$;*

$$\begin{aligned} &\| u'_{\varepsilon\mu} \|_{L^2(0,T;H)} \le C_2T/(\pi\varepsilon),\ \| u_{\varepsilon\mu} \|_{C([0,T];H)} \le C_2T\sqrt{T}/(2\pi\varepsilon), \\ &\| \varphi(u_{\varepsilon\mu}) \|_{L^\infty(0,T)} \le C\, C_2\left(C_2T/(\pi\varepsilon) + C_1\right) \text{ for all } \varepsilon > 0 \text{ and } \mu \ge 0, \end{aligned} \tag{3.10}$$

*where $C$, $C_i$, $i = 1,2$, are positive constants depending on $T$, $\| f \|_{L^2(0,T;H)}$, $\omega$, $L$, and $l$ but independent of $\varepsilon$ and $\mu$.*

**Remark 3.4** ([5]). *It is worth pointing out that all the conclusions of Theorem 3.3 remain valid under some alternative assumptions. Specifically, let us drop the condition $(H_{\varphi_c})$ but, instead, require that $B$ satisfy the stronger condition $(H_B)'$ with the Lipschitz constant $\overline{L} < \omega$. Additionally, we obtain uniqueness of the solutions to problems $(P_{\varepsilon\mu})_{ap}$ if $\varepsilon > 0$, and $(P_\mu)_{ap}$. The proof of this claim follows essentially the same arguments presented in Remark 3.1.*

**Remark 3.5** ([5]). *We mention that if $f \in L^2_{loc}(\mathbb{R};H)$, such that $f(t+T) + f(t) = 0$ for a.e. $t \in \mathbb{R}$, and $B$ is an odd operator, then the strong solutions provided by the results of this section can be extended by $T-$antiperiodicity to all of $\mathbb{R}$.*

The following section studies the behavior of solutions to problem $(P_{\varepsilon\mu})_{ap}$ with respect to $\varepsilon$ and $\mu$. We begin by assuming that $(H_A)$, $(H_{\varphi_c})$, and $(H_B)$ are satisfied. Under these assumptions, problems $(P_{\varepsilon\mu})_{ap}$ and $(P_\mu)_{ap}$ may exhibit genuine nonuniqueness of solutions (as discussed in references [8], [17], [23] and [24]). We will consider some scenarios, depending on whether the parameter $\varepsilon$ or $\mu$ tends to zero.

The convergence results (Theorems 3.4–3.7) are all established using similar techniques. The proofs rely on a combination of uniform estimates with respect to parameters $\varepsilon$ and/or $\mu$ obtained earlier, Ascoli's Lemma, the demiclosedness property of maximal monotone operators and weak convergence arguments in Sobolev spaces.

**Case 1:** Let $\varepsilon = \varepsilon_0 > 0$ be fixed and $\mu \to 0_+$.

Consider the problem $(P_{\varepsilon_0 0})_{ap} \overset{not.}{=} (P_{\varepsilon_0})_{ap}$ which corresponds to the particular case $\mu = 0$ in the problem $(P_{\varepsilon_0\mu})_{ap}$. Within this framework, the following convergence result is established.

**Theorem 3.4** ([5])**.** *Let $\varepsilon_0 > 0$ be fixed. Assume that all the assumptions of Theorem 3.1 hold with (3.1) satisfied by $\varepsilon_0$ (i.e., $L < \pi^2\varepsilon_0/T^2$). For every $\mu \geq 0$, let $u_{\varepsilon_0\mu}$ be a strong solution to problem $(P_{\varepsilon_0\mu})_{ap}$ given by Theorem 3.1. Then, for every sequence $0 < \mu_n \to 0$, there exists a subsequence, still denoted $\left(\mu_n\right)_n$, such that $u_{\varepsilon_0\mu_n} \to u$ in $C([0,T];H)$, $u_{\varepsilon_0\mu_n} \to u$ weakly in $W^{2,2}(0,T;H)$ as $n \to \infty$, and the limit $u$ is a strong solution to problem $(P_{\varepsilon_0})$. Clearly, if all the solutions under consideration are unique, then $u_{\varepsilon_0\mu} \to u$ in $C([0,T];H)$ and $u_{\varepsilon_0\mu} \to u$ weakly in $W^{2,2}(0,T;H)$ as $\mu \to 0_+$.*

**Case 2:** Let $\mu = \mu_0 > 0$ be fixed and $\varepsilon \to 0_+$.

Consider the problem derived by setting $\varepsilon = 0$ in the differential inclusion $(E_{\varepsilon\mu_0})_{ap}$, along with the $T-$antiperiodic condition. In this framework, we obtain the following convergence result.

**Theorem 3.5** ([5])**.** *Let $\mu_0 > 0$ be fixed. Assume that all the assumptions of Theorem 3.2 hold with (3.4) satisfied by $\mu_0$ (i.e., $L < \pi\mu_0/T$). For each $\varepsilon > 0$, let $u_{\varepsilon\mu_0}$ and $u_{\mu_0}$ be strong solutions to problems $(P_{\varepsilon\mu_0})_{ap}$ and $(P_{\mu_0})_{ap}$, respectively, given by Theorem 3.2. Then for every sequence $0 < \varepsilon_n \to 0$, there exists a subsequence, still denoted $\left(\varepsilon_n\right)_n$, such that $u_{\varepsilon_n\mu_0} \to u_{\mu_0}$ in $C([0,T];H)$, $u'_{\varepsilon_n\mu_0} \to u'_{\mu_0}$ weakly in $L^2(0,T;H)$ as $n \to \infty$, and the limit $u_{\mu_0}$ is a strong solution to problem $(P_{\mu_0})_{ap}$. Clearly, if all the solutions under consideration are unique, then $u_{\varepsilon\mu_0} \to u_{\mu_0}$ in $C([0,T];H)$ and $u'_{\varepsilon\mu_0} \to u'_{\mu_0}$ weakly in $L^2(0,T;H)$ as $\varepsilon \to 0_+$.*

**Case 3:** Let $\varepsilon \to \varepsilon_0$ and $\mu \to \mu_0$, with $\varepsilon_0$, $\mu_0$ satisfying $(H_{\varepsilon_0\mu_0})$

In the next result we assume that $(H_A)'$, $(H_{\varphi_c})$, and $(H_B)$ hold with constants $\omega$ and $L$ satisfying $L < \omega$. Given these assumptions, we can achieve a convergence result that is more general than those obtained in the previous two results.

**Theorem 3.6** ([5])**.** *Let $\varepsilon_0$ and $\mu_0$ be fixed, satisfying $(H_{\varepsilon_0\mu_0})$. Assume that all the assumptions of Theorem 3.3 hold. For every $\varepsilon$ and $\mu$ satisfying $(H_{\varepsilon\mu})_{ap}$, let $u_{\varepsilon\mu}$ and $u_\mu$ be strong solutions to problems $(P_{\varepsilon\mu})_{ap}$ and $(P_\mu)_{ap}$, respectively, obtained in Theorem 3.3. Then for every sequence with positive components $(\varepsilon_n, \mu_n) \to (\varepsilon_0, \mu_0)$, there exists a subsequence, still denoted $\left(\varepsilon_n, \mu_n\right)_n$,* *such that*

$$u_{\varepsilon_n\mu_n} \to u_{\mu_0} \text{ in } C([0,T];H),\ u'_{\varepsilon_n\mu_n} \to u'_{\mu_0} \text{ weakly in } L^2(0,T;H) \text{ if } \varepsilon_0 = 0,$$

$$u_{\varepsilon_n\mu_n} \to u \text{ in } C([0,T];H),\ u_{\varepsilon_n\mu_n} \to u \text{ weakly in } W^{2,2}(0,T;H) \text{ if } \varepsilon_0 > 0,$$

*and the limits $u_{\mu_0}$ and $u$ are strong solutions to problems $(P_{\mu_0})_{ap}$, and respectively $(P_{\varepsilon_0\mu_0})_{ap}$. Clearly, if all the solutions under consideration are unique, then*

$$\begin{aligned} &u_{\varepsilon\mu} \to u_{\mu_0} \text{ in } C([0,T];H),\ u'_{\varepsilon\mu} \to u'_{\mu_0} \text{ weakly in } L^2(0,T;H) \text{ if } \varepsilon_0 = 0,\\ &u_{\varepsilon\mu} \to u \text{ in } C^1([0,T];H),\ u_{\varepsilon\mu} \to u \text{ weakly in } W^{2,2}(0,T;H) \text{ if } \varepsilon_0 > 0, \end{aligned} \tag{3.11}$$

*as $(\varepsilon,\, \mu) \to (\varepsilon_0,\, \mu_0)$.*

**Theorem 3.7** ([5])**.** *Let $\varepsilon_0 \geq 0$ and $\mu_0 > 0$ be fixed. Assume that all the assumptions of Theorem 3.2 hold with (3.4) satisfied by $\mu_0$ (i.e., $L < \pi\mu_0/T$). Then there exists $\delta_0 \in (0, \mu_0)$, such that for every $\varepsilon$ and $\mu$ satisfying $(H_{\varepsilon\mu})$, with $\mid \varepsilon - \varepsilon_0 \mid < \delta_0$, $\mid \mu - \mu_0 \mid < \delta_0$, and $f \in L^2(0,T;H)$, the problems $(P_{\varepsilon\mu})_{ap}$ and $(P_\mu)_{ap}$ have at least one strong solution $u_{\varepsilon\mu} \in W^{2,2}(0,T;H)$ and $u_\mu \in W^{1,2}(0,T;H)$, respectively. Moreover, for every sequence with positive components $(\varepsilon_n, \mu_n) \to (\varepsilon_0, \mu_0)$, there exists a subsequence, still denoted $\big((\varepsilon_n, \mu_n)\big)_n$, such that*

$$\begin{aligned} &u_{\varepsilon_n\mu_n} \to u \ \text{ in } C([0,T];H),\ u_{\varepsilon_n\mu_n} \to u \ \text{ weakly in } W^{2,2}(0,T;H) \ \text{ if } \varepsilon_0 > 0, \ \text{ and}\\ &u_{\varepsilon_n\mu_n} \to u_{\mu_0} \ \text{ in } C([0,T];H),\ u'_{\varepsilon_n\mu_n} \to u'_{\mu_0} \ \text{ weakly in } L^2(0,T;H) \ \text{ if } \varepsilon_0 = 0. \end{aligned}$$

*Furthermore, the limits $u$ and $u_{\mu_0}$ are strong solutions to problems $(P_{\varepsilon_0\mu_0})_{ap}$ and $(P_{\mu_0})_{ap}$, respectively.*

**Examples** ([5]) Let $\Omega$ be a nonempty bounded domain of $\mathbb{R}^N$. The following operators on $H = L^2(\Omega)$ are odd, continuous, and sublinear, but not Lipschitz:
(1) $B\,u = \pm \mid u \mid^\alpha \operatorname{sgn} u$, with $\alpha \in (0,1)$;
(2) $B\,u = \pm \operatorname{sgn} u \cdot e^{-1/|u|}$ if $u \neq 0$, and $B\,0 = 0$.

On an arbitrary Hilbert space $H$, we can consider the operator
$B\,u = \pm \parallel u \parallel^{\alpha-1} u$, if $u \neq 0$, and $B\,0 = 0$, where $\alpha \in (0,1)$.

The following section studies the continuity of the solutions to problem $(P_{\varepsilon\mu})_{ap}$ with respect to $\varepsilon$ and $\mu$, including approximation results for the solutions to $(P_\mu)_{ap}$ and $(E_{00})_{ap}$. In what follows, we consider some specific cases where the problems $(P_{\varepsilon\mu})_{ap}$ and $(P_\mu)_{ap}$ have unique strong solutions.

In the case of uniqueness, some estimates for the convergence rate of $u_{\varepsilon\mu} - u_{\varepsilon_0\mu_0}$ if $\varepsilon_0 > 0$, and $u_{\varepsilon\mu} - u_{\mu_0}$ if $\varepsilon_0 = 0$ (in the norm of $C([0,T];H)$ or $L^2(0,T;H)$), and continuous dependence of the solution $u_{\varepsilon\mu}$ to problem $(P_{\varepsilon\mu})$ on parameters $\varepsilon$ and $\mu$ are expected.

We first begin with an investigation in the case in which the operator $A$ is strongly monotone. We will also assume $(H_A)'$, and $(H_B)'$, with constants $\omega$ and $\overline{L}$ satisfying $\overline{L} < \omega$. It is worth mentioning that assumption $(H_{\varphi_c})$ is no longer needed, so we drop it. Under these assumptions, Remark 3.4 above guarantees existence and uniqueness of the strong solutions $u_{\varepsilon\mu}$ and $u_\mu$ to problems $(P_{\varepsilon\mu})_{ap}$ and $(P_\mu)_{ap}$, respectively.

We can state the following result concerning the continuous dependence of the solution $u_{\varepsilon\mu}$ to problem $(P_{\varepsilon\mu})_{ap}$ on parameters $\varepsilon$ and $\mu$ and the approximation of the solutions of the reduced problems. The methods used to prove each of the three cases, depending on whether each parameter tends to zero individually or simultaneously, are similar and involve the use of previously deduced inequalities, properties of the subdifferential and the Arzelà-Ascoli Criterion.

**Theorem 3.8** ([5]). *Assume that $(H_A)'$ and $(H_B)'$, with constants $\omega$ and $\overline{L}$ satisfying $\overline{L} < \omega$, are fulfilled. Then for every $\varepsilon$, $\mu$ satisfying $(H_{\varepsilon\mu})$ and $f \in L^2(0,T;H)$, the problems $(P_{\varepsilon\mu})_{ap}$ and $(P_\mu)_{ap}$ have unique strong solutions $u_{\varepsilon\mu} \in W^{2,2}(0,T;H)$ and $u_\mu \in W^{1,2}(0,T;H)$, respectively. Moreover, for any fixed $\varepsilon_0$, $\mu_0$ satisfying $(H_{\varepsilon_0\mu_0})$, the following estimates and approximations hold true*

$$\begin{aligned} &\| u_{\varepsilon\mu} - u_{\varepsilon_0\mu_0} \|_{C([0,T];H)} = \mathcal{O}(| \varepsilon - \varepsilon_0 |) + \mathcal{O}(| \mu - \mu_0 |) \textit{ and} \\ &\quad u_{\varepsilon\mu} \to u_{\varepsilon_0\mu_0} \textit{ in } C^1([0,T];H) \textit{ as } (\varepsilon,\mu) \to (\varepsilon_0,\mu_0) \textit{ if } \varepsilon_0 > 0; \\ &\| u_{\varepsilon\mu} - u_{\mu_0} \|_{L^2(0,T;H)} = \mathcal{O}(\sqrt{\varepsilon}) + \mathcal{O}(| \mu - \mu_0 |) \textit{ and} \\ &\quad u_{\varepsilon\mu} \to u_{\mu_0} \textit{ in } C([0,T];H) \textit{ as } (\varepsilon,\mu) \to (0_+,\mu_0). \end{aligned} \tag{3.12}$$

*If, in addition, we assume that $B$ is an odd operator and $f \in W^{1,2}(0,T;H)$, with $f(0)+f(T) = 0$, then the (algebraic) inclusion $(E_{00})_{ap}$ admits a unique solution $u_{00} \in W^{1,2}(0,T;H)$, satisfying $u_{00}(0) + u_{00}(T) = 0$, $u(t) \in D(A)$ for all $t \in [0,T]$, and*

$$\| u_{\varepsilon\mu} - u_{00} \|_{L^2(0,T;H)} = \mathcal{O}(\sqrt{\varepsilon}) + \mathcal{O}(\mu). \tag{3.13}$$

*Furthermore, if $(h_{\varepsilon\mu})$ holds (i.e., $\mu^2/\varepsilon = \mathcal{O}(1)$), then*

$$u_{\varepsilon\mu} \to u_{00} \textit{ in } C([0,T];H) \textit{ as } (\varepsilon,\mu) \to (0_+,0_+). \tag{3.14}$$

**Examples** ([5]) Here are some examples of odd Lipschitz operators. To begin with, we consider the radial retraction on the unit ball of $H$, defined by $B_1 : H \to H$,

$$B_1 x = \begin{cases} x & \text{if } \| x \| \le 1, \\ \frac{x}{\|x\|} & \text{if } \| x \| > 1. \end{cases}$$

It is well known that $B_1$ is Lipschitz continuous with a Lipschitz constant $L_1 = 1$.
Another example (as shown in [24, Lemma 2.3]) is the operator $B_2 : H \to H$,

$$B_2 x = \begin{cases} x & \text{if } \| x \| \le 1, \\ \frac{x}{\|x\|^2} & \text{if } \| x \| > 1, \end{cases}$$

which also has a Lipschitz constant equal to 1.

Let us now consider the specific case where $H = L^2(\Omega)$, with $\Omega$ being a nonempty bounded domain in $\mathbb{R}^N$. Some examples of odd Lipschitz operators on $H$ are given by: (1) $B\,u = \pm \sin u$; (2) $B(u) = \pm\, u/\sqrt{1+u^2}$; (3) $B\,u = \pm\,\mathrm{sgn}\, u \cdot \min\{|\, u\,|^\alpha, |\, u\,|^\beta\}$, with $0 < \alpha < 1 < \beta$.

Next, we provide some sufficient conditions that guarantee the uniqueness of strong solutions to problems $(P_{\varepsilon\mu})_{ap}$ and $(P_\mu)_{ap}$, applicable when the operator $A$ is no longer strongly monotone.

**Theorem 3.9** ([5]). *Let $\varepsilon_0$ and $\mu_0$ be fixed, satisfying $(H_{\varepsilon_0\mu_0})$. Assume that $(H_A)$ and $(H_B)'$ hold, with Lipschitz constant $\overline{L}$ of $B$ satisfying*

$$\overline{L} < \frac{\pi^2 \varepsilon_0}{T^2} \quad \text{if } \varepsilon_0 > 0 \quad \text{and} \quad \overline{L} < \frac{\pi \mu_0}{T} \quad \text{if } \varepsilon_0 = 0.$$

*In addition to the assumptions stated above, if $\varepsilon_0 = 0$, suppose that $A$ is a single valued, linear operator and that $(H_{\varphi_c})$ holds.*

*Then there exists $\delta_0 \in (0, \max\{\varepsilon_0, \mu_0\})$, such that for every $\varepsilon$ and $\mu$ satisfying $(H_{\varepsilon\mu})$, with $|\ \varepsilon - \varepsilon_0\ | < \delta_0$, $|\ \mu - \mu_0\ | < \delta_0$, and $f \in L^2(0,T;H)$, the problems $(P_{\varepsilon\mu})_{ap}$ and $(P_\mu)_{ap}$ have unique strong solutions, $u_{\varepsilon\mu} \in W^{2,2}(0,T;H)$ and $u_\mu \in W^{1,2}(0,T;H)$, respectively. Moreover, the following estimate and approximations hold true*

$$\begin{aligned}
&\| \ u_{\varepsilon\mu} - u_{\varepsilon_0\mu_0} \ \|_{C([0,T];H)} = \mathcal{O}(|\ \varepsilon - \varepsilon_0\ |) + \mathcal{O}(|\ \mu - \mu_0\ |) \quad \text{and} \\
&\ u_{\varepsilon\mu} \to u_{\varepsilon_0\mu_0} \ \text{in } C^1([0,T];H) \ \text{ as } (\varepsilon,\mu) \to (\varepsilon_0,\mu_0) \ \text{if } \varepsilon_0 > 0; \\
&\ u_{\varepsilon\mu} \to u_{\mu_0} \ \text{in } C([0,T];H) \ \text{ and} \\
&\ u'_{\varepsilon\mu} \to u'_{\mu_0} \ \text{weakly in } L^2(0,T;H) \ \text{ as } (\varepsilon,\mu) \to (0_+,\mu_0).
\end{aligned} \tag{3.15}$$

The final section is devoted to illustrating the abstract results through concrete applications, including semilinear and nonlinear problems subject to antiperiodic conditions in time.

# Chapter 4

## A Class of Antiperiodic Boundary Value Problems Governed by Maximal Monotone Operators

The results presented in this chapter are part of an article published in *An. Şt. Univ. Ovidius Constanţa* [25].

**Among the original contributions in this chapter,** we mention Theorems 4.1 and 4.2.

Building on the previous chapters, we now examine second- and first-order inclusions with antiperiodic boundary conditions, where $A$ is an odd strongly maximal monotone operator. This broader setting allows for applications to nonlinear models like hyperbolic systems. Since $A$ is not a subdifferential, stronger conditions on $B$ and $f$ are required to ensure well-posedness.

Recall that $H$ denotes a real Hilbert space with inner product $(\cdot,\cdot)$ and norm $\|\cdot\|$.

We consider in a real Hilbert space $H$ the same class of problems as the previous chapter

$$(P_{\varepsilon\mu})_{ap} \quad \begin{cases} -\varepsilon u''(t) + \mu u'(t) + Au(t) + Bu(t) \ni f(t) \quad \text{a.e. in } (0,T), & (E_{\varepsilon\mu}) \\ u(0) + u(T) = 0, \quad u'(0) + u'(T) = 0, \end{cases}$$

where $T > 0$, $\varepsilon > 0$, and $\mu \geq 0$, along with

$$(P_{\mu})_{ap} \quad \begin{cases} \mu u'(t) + Au(t) + Bu(t) \ni f(t) \quad \text{a.e. in } (0,T), & (E_{\mu}) \\ u(0) + u(T) = 0. \end{cases}$$

for $\mu > 0$. We also introduce the limiting algebraic inclusion

$$(E_{00})_{ap} \quad Au(t) + Bu(t) \ni f(t) \quad \text{a.e. in } (0,T),$$

which is obtained by formally setting $\varepsilon = \mu = 0$ in $(E_{\varepsilon\mu})$.

The analysis is carried out under the following assumptions on the data and operators involved in problems $(P_{\varepsilon\mu})_{ap}$, $(P_\mu)_{ap}$, and $(E_{00})_{ap}$.

$(H_f)$ $f \in W^{1,2}(0,T;H)$ and $f(0)+f(T)=0$;

$(H_A)$ The operator $A : D(A) \subset H \to H$ is odd, strongly maximal monotone (possibly set-valued), with constant $\omega_0 > 0$;

$(H_B)$ The operator $B : H \to H$ is odd, maximal monotone (possibly set-valued) and satisfies the following condition: for each $r > 0$, there is $L_r > 0$ with the property that for all $x \in H$ with $\| x \| \leq r$, it holds that $\| B\, x \| \leq L_r$.

**Remark 4.1.** *A typical example of an operator $B$ satisfying assumption $(H_B)$ is given by $B\, x = \| x \|^{p-2}\, x$ for all $x \in H$. This operator is also cyclically monotone, as it corresponds to the subdifferential of the convex function $x \to \| x \|^p$ .*

We begin by proving that the problems introduced above admit unique solutions. Additionally, we obtain some uniform estimates with respect to the parameters $\varepsilon$ and $\mu$ of these solutions. These estimates will be crucial in proving the results presented in the subsequent sections.

Throughout this chapter, all solutions to the aforementioned problems are considered in the sense of Definition 1.

**Theorem 4.1** ([25])**.** *$(i)$ Assume that $A$ is an odd maximal monotone operator and $(H_B)$ is fulfilled. Then, for every $\varepsilon > 0$, $\mu \geq 0$, and $f \in L^2(0,T;H)$, the problem $(P_{\varepsilon\mu})_{ap}$ has a unique solution $u_{\varepsilon\mu} \in W^{2,2}(0,T;H)$ which satisfies the following estimate*

$$\varepsilon \, \| u''_{\varepsilon\mu} \|_{L^2(0,T;H)} \leq \| f \|_{L^2(0,T;H)} \, . \tag{4.1}$$

*$(ii)$ Assume that $(H_A)$ is satisfied. Then, for every nonnegative $\varepsilon$ and $\mu$ such that $\varepsilon + \mu > 0$, and for $f$ satisfying $(H_f)$, both problems $(P_\mu)_{ap}$ and $(P_{\varepsilon\mu})_{ap}$ have unique solutions $u_\mu \in W^{1,2}(0,T;H)$ and $u_{\varepsilon\mu} \in W^{2,2}(0,T;H)$, respectively. Moreover, the following estimates hold*

$$\begin{aligned} \| u'_\mu \|_{L^2(0,T;H)} &\leq \omega_0^{-1} \, \| f' \|_{L^2(0,T;H)} \quad \text{for every } \mu > 0, \\ \| u'_{\varepsilon\mu} \|_{L^2(0,T;H)} &\leq \omega_0^{-1} \, \| f' \|_{L^2(0,T;H)} \quad \text{for every } \varepsilon > 0, \ \mu \geq 0. \end{aligned} \tag{4.2}$$

*In addition, the algebraic inclusion $(E_{00})_{ap}$ has a unique solution $u_{00} \in W^{1,2}(0,T;H)$, satisfying $u_{00}(0) + u_{00}(T) = 0$ and $u(t) \in D(A)$ for all $t \in [0,T]$.*

The proof is based on specific techniques of maximal monotone operators, which include using the Yosida approximation, the demiclosedness property of canonical extensions of operators $A$ and $B$, as well as the Arzelà–Ascoli Criterion.

The following section is designated to investigate the continuous dependence of the solution $u_{\varepsilon\mu}$ to problem $(P_{\varepsilon\mu})_{ap}$ on parameters $\varepsilon$ and $\mu$. We also obtain approximating results regarding the solutions to the reduced problem $(P_\mu)_{ap}$ and the algebraic inclusion $(E_{00})_{ap}$.

**Theorem 4.2** ([25]). *Assume that* $(H_B)$ *is fulfilled.*
*(i) Let* $\varepsilon_0 > 0$ *and* $\mu_0 \geq 0$ *be fixed. Suppose that* $A$ *is an odd maximal monotone operator. For every* $\varepsilon > 0$, $\mu \geq 0$, *and* $f \in L^2(0,T;H)$, *let* $u_{\varepsilon\mu} \in W^{2,2}(0,T;H)$ *be the unique solution to problem* $(P_{\varepsilon\mu})_{ap}$ *given by Theorem 4.1 (i). Then the following estimate and convergence hold*

$$\begin{aligned} &\| u_{\varepsilon\mu} - u_{\varepsilon_0\mu_0} \|_{C([0,T];H)} = \mathcal{O}(| \varepsilon - \varepsilon_0 |) + \mathcal{O}(| \mu - \mu_0 |), \\ &u_{\varepsilon\mu} \to u_{\varepsilon_0\mu_0} \text{ in } C^1([0,T];H) \text{ as } (\varepsilon,\mu) \to (\varepsilon_0,\mu_0). \end{aligned} \tag{4.3}$$

*(ii) Let* $\mu_0 > 0$ *be fixed. Assume that* $(H_A)$ *holds. For every nonnegative* $\varepsilon$ *and* $\mu$ *such that* $\varepsilon + \mu > 0$, *and* $f$ *satisfying* $(H_f)$, *let* $u_{\varepsilon\mu} \in W^{2,2}(0,T;H)$ *and* $u_\mu \in W^{1,2}(0,T;H)$ *be the unique solutions to problems* $(P_{\varepsilon\mu})_{ap}$, *and respectively* $(P_\mu)_{ap}$, *given by Theorem 4.1 (ii). Then, the following estimate and approximation hold*

$$\begin{aligned} &\| u_{\varepsilon\mu} - u_{\mu_0} \|_{L^2(0,T;H)} = \mathcal{O}(\sqrt{\varepsilon}) + \mathcal{O}(| \mu - \mu_0 |), \\ &u_{\varepsilon\mu} \to u_{\mu_0} \text{ in } C([0,T];H) \text{ as } (\varepsilon,\mu) \to (0_+,\mu_0). \end{aligned} \tag{4.4}$$

*Moreover, the following estimate is also valid*

$$\| u_{\varepsilon\mu} - u_{00} \|_{L^2(0,T;H)} = \mathcal{O}(\sqrt{\varepsilon}) + \mathcal{O}(\mu) \text{ as } (\varepsilon,\mu) \to (0_+,0_+), \tag{4.5}$$

*where* $u_{00} \in W^{1,2}(0,T;H)$ *is the unique solution to the (algebraic) inclusion* $(E_{00})_{ap}$, *given by Theorem 4.1(ii). Furthermore, if* $\mu^2/\varepsilon = \mathcal{O}(1)$, *then*

$$u_{\varepsilon\mu} \to u_{00} \text{ in } C([0,T];H) \text{ as } (\varepsilon,\mu) \to (0_+,0_+). \tag{4.6}$$

Finally, the last section applies the abstract work to an antiperiodic semilinear telegraph system.

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