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Doctoral Thesis

Summary

Contributions to the Study of Eigenvalue Problems for Nonlinear Elliptic Operators

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Introduction

The p-Laplacian operator, also known as the p-harmonic operator and denoted by Δ_p, represents one of the most studied operators in the theory of partial differential equations. Defined for $p \in (1,\infty)$ by

$$\Delta_p u := \operatorname{div}\left(|\nabla u|^{p-2}\nabla u\right),$$

this operator is nonlinear for any $p \neq 2$, while in the particular case $p = 2$, it coincides with the Laplace operator.

Its importance is justified by numerous applications in physics and engineering: modeling flows in porous media (the nonlinear Darcy law), behavior of plastic materials, nonlinear heat transfer, glacier dynamics, or the description of Brownian motion. For details, see Benedikt et al. [15], Lindqvist [29], and Barbu, Rehmeier and Röckner [10].

To introduce the classical eigenvalue problems associated with the p-Laplacian operator, we consider a bounded domain $\Omega \subset \mathbb{R}^N$, with $N \geq 2$, having smooth boundary $\partial\Omega$. The eigenvalue problem associated with the operator $-\Delta_p$ with Dirichlet boundary conditions is:

$$(P_D^0): \begin{cases} -\Delta_p u &= \lambda |u|^{p-2}u \quad \text{in } \Omega, \\ u &= 0 \quad \text{on } \partial\Omega. \end{cases}$$

A real number λ is an eigenvalue of this problem if there exists $u_\lambda \in W_0^{1,p}(\Omega) \setminus \{0\}$ such that

$$\int_\Omega |\nabla u_\lambda|^{p-2}\nabla u_\lambda \cdot \nabla w \, dx = \lambda \int_\Omega |u_\lambda|^{p-2}u_\lambda w \, dx, \quad \forall w \in W_0^{1,p}(\Omega).$$

It is known that problem (P_D^0) admits a sequence of eigenpairs (λ_n, u_n), with

$$0 < \lambda_1 < \lambda_2 \leq \lambda_3 \leq \cdots \to \infty,$$

obtained by variational methods, the most well-known being based on the Krasnosel'skiĭ genus and the Lusternik–Schnirelmann Principle (see Gasinski and Papageorgiou [26, Section 6.2]).

Except for the cases $p = 2$ or $N = 1$, it is unknown whether the spectrum consists solely of these eigenvalues (see Gasinski and Papageorgiou [26, Sections 6.1 and 6.3]). Thus, the spectrum of the operator $-\Delta_p$, defined on the Sobolev space $W_0^{1,p}(\Omega)$, represents an open problem for decades, except for the first eigenvalue λ_1 (called the principal eigenvalue). This eigenvalue is simple and has an associated eigenfunction strictly positive in Ω (see Lê [28, Theorem 5.1]). Moreover, λ_1 can be variationally characterized as the minimum of the Rayleigh quotient, that is,

$$\lambda_1 = \min_{u \in W_0^{1,p}(\Omega)} \frac{\int_\Omega |\nabla u|^p}{\int_\Omega |u|^p},$$

(see Motreanu et al. [34, Proposition 9.6]).

Similar results also exist for other types of boundary conditions, such as Neumann, Robin, or Steklov (see Lê [28]).

The operator $-(\Delta_p + \Delta_q)$, with $p, q \in (1,\infty)$, $p \neq q$, also known as the $-(p,q)$-Laplacian operator, represents a perturbation of the operator $-\Delta_p$. Unlike $-\Delta_p$, this operator is nonhomogeneous. Consequently, the application of variational techniques leads, in certain cases, to the determination of the entire set of eigenvalues. We will refer to these results in the following sections.

The $-(p, q)$-Laplacian operator is a combination of two nonlinear diffusions of different orders, reflecting the interaction of two transfer mechanisms with distinct regimes. Due to this structure, it has numerous applications in mathematical physics. For example, in the case when $p = 2$ and $q > 1$, the operator $\Delta + a_q\Delta_q$, with $a_q > 0$, appears in the Born–Infeld theory for electrostatic fields (see Bonheure, Colasuonno, and Fortunato [16], as well as Fortunato, Orsina and Pisani [24]). Other applications of the (p, q)-Laplacian operator can be found in quantum physics (see Benci et al. [13] and Benci, Fortunato and Pisani [14]), in reaction–diffusion systems (see Cherfils and Il'yasov [17]), and also in nonlinear elasticity theory (see Marcellini [31] and Zhikov [39]).

Two-phase eigenvalue problems are also motivated by models arising from classical relativity. An example in this regard is the operator

$$Qu := -\operatorname{div}\left(\frac{\nabla u}{\sqrt{1 - |\nabla u|^2}}\right),$$

which appears as the mean curvature operator in the Lorentz–Minkowski space (see Bartnik and Simon [11]). A first-order approximation of this operator is

$$-\Delta u - \Delta_4 u,$$

which corresponds to the $-(2, 4)$-Laplacian operator (see Pompio and Watanabe [38]).

Given the vast applicability of the (p, q)-Laplacian operator, the literature dedicated to it, including the study of associated eigenvalue problems, is already extensive and continues to develop. We mention, in this regard, two relevant survey papers by Marano and Mosconi [30] and by Barbu and Moroșanu [5].

Literature Review

To motivate the topic addressed in this thesis and to highlight the obtained contributions, we briefly present the three problems that will be analyzed in the following chapters. For simplicity, in this chapter only, we denote them by (P_i), $i = \overline{1, 3}$.

Consider in what follows a bounded domain $\Omega \subset \mathbb{R}^N$, with $N \geq 2$, having a smooth boundary $\partial\Omega$.

The first problem investigated is an eigenvalue problem defined in Ω, which also includes potentials with nonnegative weights of order p and q, as well as parametric boundary conditions:

$$(P_1): \begin{cases} -(\Delta_p u + \Delta_q u) + \rho_1(x)|u|^{p-2}u + \rho_2(x)|u|^{q-2}u = \lambda\alpha(x)|u|^{r-2}u, & x \in \Omega, \\ \dfrac{\partial u}{\partial \nu_{pq}} + \gamma_1(x)|u|^{p-2}u + \gamma_2(x)|u|^{q-2}u = \lambda\beta(x)|u|^{r-2}u, & x \in \partial\Omega, \end{cases}$$

where $p, q, r \in (1, \infty)$ with $p < q$ and $\alpha, \rho_i \in L^\infty(\Omega)$, $\beta, \gamma_i \in L^\infty(\partial\Omega)$ are nonnegative functions satisfying

$$(h_{\alpha\beta}): \int_\Omega \alpha\, dx + \int_{\partial\Omega} \beta\, d\sigma > 0, \quad (h_{\rho\gamma}): \int_\Omega \rho_i\, dx + \int_{\partial\Omega} \gamma_i\, d\sigma > 0, \quad i = 1, 2.$$

In the considered boundary condition, we used the notation

$$\frac{\partial u}{\partial \nu_{pq}} := \left(|\nabla u|^{p-2} + |\nabla u|^{q-2}\right)\frac{\partial u}{\partial \nu}, \tag{1}$$

where ν is the outward unit normal to $\partial\Omega$. This notation will also be used in what follows. The second problem has a similar structure, but contains only potentials with nonnegative weights of order q, so there is no longer symmetry between the exponents p and q:

$$(P_2): \begin{cases} -(\Delta_p u + \Delta_q u) + \rho(x)|u|^{q-2}u = \lambda\alpha(x)|u|^{r-2}u, & x \in \Omega, \\ \dfrac{\partial u}{\partial \nu_{pq}} + \gamma(x)|u|^{q-2}u = \lambda\beta(x)|u|^{r-2}u, & x \in \partial\Omega, \end{cases}$$

where $1 < q < r < p < \infty$ and the functions $\alpha, \rho \in L^\infty(\Omega)$, $\beta, \gamma \in L^\infty(\partial\Omega)$ are nonnegative and satisfy the hypotheses $(h_{\alpha\beta})$, $(h_{\rho\gamma})$ introduced earlier in the presentation of problem (P_1).

The last problem studied is a nonlinear transmission eigenvalue problem, defined as follows:

$$(P_3): \begin{cases} -\Delta_p u_1 + \gamma_1(x)|u_1|^{r-2}u_1 = \lambda|u_1|^{p-2}u_1, & \text{in } \Omega_1, \\ -\Delta_q u_2 + \gamma_2(x)|u_2|^{s-2}u_2 = \lambda|u_2|^{q-2}u_2, & \text{in } \Omega_2, \\ u_1 = u_2, \quad \dfrac{\partial u_1}{\partial \nu_p} = \dfrac{\partial u_2}{\partial \nu_q}, & \text{on } \Gamma, \\ \dfrac{\partial u_2}{\partial \nu} + \beta(x)|u_2|^{\zeta-2}u_2 = 0, & \text{on } \Sigma, \end{cases}$$

where Ω_1 is a subdomain with smooth boundary Γ, with $\overline{\Omega}_1 \subset \Omega$, and $\Omega_2 := \Omega \setminus \overline{\Omega}_1$. It is assumed that $p, q, r, s, \zeta \in (1, \infty)$, $\gamma_i \in L^\infty(\Omega_i)$ for $i = 1, 2$, and $\beta \in L^\infty(\Sigma)$ is nonnegative a.e. on Σ. In the boundary conditions on Γ and Σ, we have denoted

$$\frac{\partial u_i}{\partial \nu_\theta} := |\nabla u_i|^{\theta-2}\nabla u_i \cdot \nu_\theta, \quad \theta \in \{p, q\}, \quad i \in \{1, 2\}, \quad \frac{\partial u_2}{\partial \nu} := |\nabla u_2|^{q-2}\nabla u_2 \cdot \nu,$$

where $\nu_p + \nu_q = 0$ are the outward unit normal on Γ, and ν is the the outward unit normal on Σ.

Additional hypotheses concerning the exponents and coefficients will be specified in the chapters dedicated to each problem.

We start with the case of Neumann-type boundary conditions. In this respect, we introduce the eigenvalue problem

$$(P_N): \begin{cases} -(\Delta_p u + \Delta_q u) = \lambda|u|^{q-2}u & \text{in } \Omega, \\ \frac{\partial u}{\partial \nu_{pq}} = 0 \quad \text{on } \partial\Omega. \end{cases}$$

A real number λ is an eigenvalue for (P_N) if there exists $u_\lambda \in W := W^{1,\max\{p,q\}}(\Omega) \setminus \{0\}$ such that

$$\int_\Omega \left(|\nabla u_\lambda|^{p-2} + |\nabla u_\lambda|^{q-2}\right) \nabla u_\lambda \cdot \nabla u \, dx = \lambda \int_\Omega |u_\lambda|^{q-2}u_\lambda u \, dx \quad \forall u \in W.$$

In this case, the function u_λ is called an *eigenfunction* corresponding to the eigenvalue λ, and the pair (λ, u_λ) is called an *eigenpair* of the problem (P_N).

For $p > 2$ and $q = 2$, Mihăilescu [32, Theorem 1.1] showed that the spectrum of problem (P_N) is $\{0\} \cup (\lambda^N(p, 2), \infty)$, where

$$\lambda^N(p, 2) := \left\{ \inf_{u \in W \setminus \{0\}} \frac{\int_\Omega |\nabla u|^2 \, dx}{\int_\Omega u^2 \, dx}; \quad \int_\Omega u \, dx = 0 \right\} > 0.$$

For $p < 2$, Fărcășeanu et al. [19, Theorem 1.1] identified the entire set of eigenvalues for (P_N) as the set $\{0\} \cup (\lambda^N(p, 2), \infty)$. Mihăilescu and Moroșanu [33] treated the general case $p \in (1, \infty)$,

$q > 2$, obtaining the spectrum $\{0\} \cup (\lambda^N(p,q), \infty)$, where

$$\lambda^N(p,q) := \left\{ \inf_{u \in W \setminus \{0\}} \frac{\int_\Omega |\nabla u|^q \, dx}{\int_\Omega |u|^q \, dx}; \quad \int_\Omega |u|^{q-2} u \, dx = 0 \right\} > 0. \tag{2}$$

Let us consider the eigenvalue problem associated with the Steklov (p,q)-Laplacian operator

$$(P_S): \begin{cases} -(\Delta_p u + \Delta_q u) = 0 & \text{in } \Omega, \\ \dfrac{\partial u}{\partial \nu_{pq}} = \lambda |u|^{q-2} u & \text{on } \partial\Omega. \end{cases}$$

Costea and Moroșanu [18, Theorem 3.1] for the case $p \in (1,\infty), q \in [2,\infty)$ determined the spectrum of problem (P_S) as the set $\{0\} \cup (\lambda^S(p,q), \infty)$, where

$$\lambda^S(p,q) := \left\{ \inf_{u \in W \setminus \{0\}} \frac{\int_\Omega |\nabla u|^q \, dx}{\int_{\partial\Omega} |u|^q \, d\sigma}; \quad \int_{\partial\Omega} |u|^{q-2} u \, d\sigma = 0 \right\} > 0.$$

In the case of the Robin (p,q)-Laplacian operator, we have the following eigenvalue problem

$$(P_R): \begin{cases} -(\Delta_p u + \Delta_q u) = \lambda |u|^{q-2} u & \text{in } \Omega, \\ \dfrac{\partial u}{\partial \nu_{pq}} + \beta |u|^{q-2} u = 0 & \text{on } \partial\Omega, \end{cases}$$

where $p, q \in (1,\infty)$, $p \neq q$, and β is a positive constant.

Problem (P_R) was studied by Gyulov and Moroșanu [27], who determined an interval of eigenvalues $(\lambda^R(p,q), \lambda_0)$ and moreover proved that there are no eigenvalues in the interval $(-\infty, \lambda^R(p,q)]$.

The above constants are positive and defined by

$$\lambda^R(p,q) := \inf_{u \in W \setminus \{0\}} \frac{\int_\Omega |\nabla u|^q \, dx + \beta \int_{\partial\Omega} |u|^q \, d\sigma}{\int_\Omega |u|^q \, dx} < \lambda_0 := \beta \frac{\int_{\partial\Omega} d\sigma}{\int_\Omega dx}. \tag{3}$$

The authors stated as an open problem the existence of eigenvalues in the interval $[\lambda_0, \infty)$.

We also mention the work of Papageorgiou et al. [36], where a more general eigenvalue problem than (P_R) is analyzed for the case $1 < p < q$. In this work, the operator $-(\Delta_p + \Delta_q)$ is perturbed by a q-order potential with an indefinite weight $\zeta \in L^s(\Omega)$, where $s < \frac{N}{q}$ if $q \leq N$ and $s = 1$ if $q > N$. The constant β is replaced by a function $\beta \in W^{1,\infty}(\partial\Omega)$, $\beta \geq 0$, $\beta \not\equiv 0$, satisfying the condition

$$\int_\Omega \zeta \, dx + \int_{\partial\Omega} \beta \, d\sigma > 0.$$

Using an approach similar to that in [27], the authors obtain a comparable result (see [36, Theorem 1]).

The study of eigenvalue problems with boundary conditions of the type considered in problems (P_1) and (P_2) was initiated by Von Below and François [12], in the particular case of the Laplace operator for $\alpha = 1$ and $\beta > 0$ a continuous function on $\partial\Omega$.

This problem is known in the literature as the dynamic type eigenvalue problem, because it appears in the study of parabolic equations with dynamic boundary conditions (see [25]).

Starting from this linear model, subsequent research focused on extending it to nonlinear contexts with operators of p-Laplacian or (p,q)-Laplacian type.

In this context, we consider the following generalized eigenvalue problem:

$$(P_{\text{gen}}): \quad \begin{cases} -(\Delta_p u + \Delta_q u) = \lambda\, \alpha(x)\, |u|^{r-2} u & \text{in } \Omega, \\ \dfrac{\partial u}{\partial \nu_{pq}} = \lambda\, \beta(x)\, |u|^{r-2} u & \text{on } \partial\Omega, \end{cases}$$

where $p, q, r \in (1, \infty)$ with $p \neq q$, and α, β are nonnegative functions satisfying the hypothesis $(h_{\alpha\beta})$ formulated within problem (P_1).

It is well known that the eigenfunctions of problem (P_{gen}) belong to the set

$$\mathcal{C}_r := \left\{ u \in W; \quad \int_\Omega \alpha\, |u|^{r-2} u\, dx + \int_{\partial\Omega} \beta\, |u|^{r-2} u\, d\sigma = 0 \right\}.$$

In the case $r = q$, Barbu and Moroșanu [7, Theorem 1] showed that the set of eigenvalues of problem (P_{gen}) is equal to $\{0\} \cup (\widetilde{\lambda}(p, q), \infty)$, where

$$0 < \widetilde{\lambda}(p, q) := \inf_{u \in \mathcal{C}_q \setminus \{0\}} \frac{\int_\Omega |\nabla u|^q\, dx}{\int_\Omega \alpha\, |u|^q\, dx + \int_{\partial\Omega} \beta\, |u|^q\, d\sigma}.$$

This result generalizes both the previous cases obtained for problems (P_N) and (P_S), as well as the result obtained by Abreu and Madeira [1] for problem (P_{gen}) with $q = 2$ and $p \in (1, \infty)$, $p \neq 2$.

If $r \neq q$, we assume, without loss of generality, that $1 < p < q$. Barbu and Moroșanu [8] proved that if either $1 < r < p < q < \infty$, or $1 < q < p < r < \infty$ and $r \in \left(1, \frac{q(N-1)}{N-q}\right)$ for $q < N$, then the set of eigenvalues of problem (P_{gen}) is $[0, \infty)$.

On the other hand, in [6], the same authors showed that if $1 < p < r < q < \infty$, with $r < \frac{q(N-1)}{N-q}$ for $q < N$, then there exist two strictly positive constants, $0 < \lambda_* < \lambda^*$, such that any $\lambda \in \{0\} \cup [\lambda^*, \infty)$ is an eigenvalue of (P_{gen}), while the same problem has no eigenvalues in $\lambda \in (-\infty, \lambda_*) \setminus \{0\}$.

In what follows, we will highlight the **original results obtained within this thesis** for the first two problems, (P_1) and (P_2), which extend or generalize the contributions already existing in the literature.
More precisely, in the works of Barbu, Burlacu, and Moroșanu [2, 4], where the previously introduced problems (P_1) and (P_2) were studied, we generalized and/or extended the results obtained for problems (P_N) and (P_S) in the works [18, 19, 27, 32, 33, 36]. Indeed, by choosing the functions α or β equal to zero or one, respectively, one recovers the Neumann or Steklov type boundary conditions.

Regarding the results obtained in [6, 7, 8], these were extended in [4] by studying the spectrum of problem (P_2), where we introduced potentials of order q in the equation and/or on the boundary. Using Krasnosel'skii genus and the Lusternik–Schnirelmann Principle, we gave a positive answer to the open problem from [27], showing that problem (P_R) has eigenvalues greater than λ_0.

On the other hand, the introduction of potentials of order p in problem (P_1), alongside those of order q, led to the conclusion that, in the case of problem (P_R), the presence of a potential of order p in the equation ensures a complete characterization of its spectrum. Thus, if we choose

$$r = q, \quad \alpha = 1, \quad \rho_2 = 0 \text{ in } \Omega, \quad \beta = \gamma_1 = 0, \quad \gamma_2 = \text{const.} > 0 \text{ on } \partial\Omega,$$

and the weight $\rho_1 \geq 0$ a.e. in Ω, with $\int_\Omega \rho_1(x)\, dx > 0$, then, according to Theorem 2.1.1(b), the spectrum of the problem is exactly the interval $(\widehat{\lambda}_q, \infty)$, where $\widehat{\lambda}_q > 0$ is defined as in formula (3), with γ_2 instead of β, according to the notations.

The obtained results show that not only the perturbation of the operator $-\Delta_q$ by $-\Delta_p$, but also the perturbation of the equation or of the boundary conditions with potential-type terms of order p can lead to obtain a continuous spectrum for the problem, while in the absence of such perturbations there are no results that fully characterize this spectrum.

Next, we present recent work dedicated to nonlinear transmission problems, similar to problem (P_3). Transmission problems have various applications in fluid mechanics, physics, chemistry, biology, hence the importance of their study (see Fife [20], Nicaise [35]).
Recall, for example, that Figueiredo and Montenegro [21] investigated a transmission problem with critical exponential growth, more precisely, the nonlinearities behave like $\exp(\alpha_0 s^2)$ as $|s| \to \infty$, for a constant $\alpha_0 > 0$. The authors proved that the following elliptic transmission problem in $\mathbb{R}^2$

$$\begin{cases} -\Delta u_1 = f(x, u_1) & \text{in } \Omega_1, \\ -\Delta u_2 = g(x, u_2) & \text{in } \Omega_2, \\ u_1 = u_2, \quad \frac{\partial u_1}{\partial \nu_1} = \frac{\partial u_2}{\partial \nu_2} & \text{on } \Gamma, \\ u_2 = 0 \quad \text{on } \Sigma, \end{cases}$$

has a nontrivial solution.

Also, the transmission problem with critical growth

$$\begin{cases} -\Delta u_1 = \lambda f(x, u_1) & \text{in } \Omega_1, \\ -\Delta u_2 = |u_2|^{2^*-2} u_2 & \text{in } \Omega_2, \\ u_1 = u_2, \quad \frac{\partial u_1}{\partial \nu_1} = \frac{\partial u_2}{\partial \nu_2} & \text{on } \Gamma, \\ u_2 = 0 \quad \text{on } \Sigma, \end{cases}$$

was studied by the same authors in [22]. They showed that for sufficiently large λ, the problem admits a nontrivial solution. Other existing results for nonlinear transmission problems approached by variational methods can be found in [23, 33].

A similar problem to (P_3) was investigated by Barbu et al. in [9]. The authors considered a bounded domain $\Omega \subset \mathbb{R}^N$, $N \geq 2$, with a Lipschitz boundary $\partial\Omega$, which is divided into two Lipschitz subdomains, Ω_1 and Ω_2. In other words, $\Omega = \Omega_1 \cup \Omega_2 \cup \Gamma$. It is assumed that the boundary $\partial\Omega$ is split into two parts, $\partial\Omega_1$ and $\partial\Omega_2$, such that $\partial\Omega_1 = \Gamma_1 \cup \Gamma$ and $\partial\Omega_2 = \Gamma_2 \cup \Gamma$. In this setting, the following eigenvalue problem was considered:

$$\begin{cases} -\Delta_p u_1 = \lambda\, |u_1|^{p-2} u_1 & \text{in } \Omega_1, \\ -\Delta_q u_2 = \lambda\, |u_2|^{q-2} u_2 & \text{in } \Omega_2, \\ \dfrac{\partial u_1}{\partial \nu_p} = 0 \quad \text{on } \Gamma_1, & \dfrac{\partial u_2}{\partial \nu_q} = 0 \quad \text{on } \Gamma_2, \\ u_1 = u_2, \quad \dfrac{\partial u_1}{\partial \nu_p} = \dfrac{\partial u_2}{\partial \nu_q} & \text{on } \Gamma, \end{cases} \tag{0.1}$$

where on the boundary, $\frac{\partial u}{\partial \nu_r}$, $r = p, q$, denotes the conormal derivatives of the operators involved in the problem, similarly to those in the formulation of problem (P_3). Using the Lusternik–Schnirelmann Principle, the authors proved the existence of a sequence of eigenvalues of the above problem tending to infinity.

The nonlinear transmission problem (P_3), investigated by us in the work Barbu, Burlacu and Moroșanu [3], generalizes this result by including undefined potentials in the two subdomains, which are configured differently from the case presented above. Moreover, if $\beta = 0$, then the conditions on Σ become of Neumann type. Furthermore, using similar arguments, cases with generalized Dirichlet or Neumann boundary conditions, as well as different partitions of the domain (including those of the type considered in [9]), can also be studied.

Motivation and Objectives of the Thesis

The study of eigenvalues for nonlinear operators of the (p, q)-Laplacian has significant development in recent decades, both theoretically and in the context of applications.

The motivation of this thesis is based on the need to extend the current framework of the theory, especially in the following directions:
(i) the simultaneous consideration of two nonlinear diffusion operators of different orders, of p- and q-Laplacian with $p \neq q$;
(ii) the inclusion of potentials with nonnegative weights in the equation and/or on the boundary;
(iii) the treatment of generalized boundary conditions, in which the parameter appears both in the equation and in the boundary condition;
(iv) the analysis of a transmission problem between two subdomains governed by different differential operators.

Based on these general directions, the **objectives of the thesis** are the following:

1 The study of an eigenvalue problem with parametric boundary conditions, in which potentials with nonnegative weights of order p and q appear, both in the equation and on the boundary. This problem, denoted by (P_1), extends the classical Neumann and Steklov cases;

2 The investigation of an asymmetric version, (P_2), in which only the potentials of order q are present, removing the symmetry between the components of order p and q. The aim is to characterize the spectrum depending on the relative positioning of p, q, and r;

3 The analysis of a transmission problem (P_3), in which the domain is divided into two disjoint subdomains Ω_1 and Ω_2, with p- and q-Laplacian operators acting separately, connected through continuity conditions on the solution and the flux on the common interface Γ;

4 The proof of the existence of sequences of eigenvalues tending to infinity and, in some cases, the complete characterization of the spectrum depending on the involved parameters;

5 The extension of recent results from the literature by using the following variational methods: direct methods, the fibering method, min-max methods based on Krasnosel'skiĭ genus and the Lusternik-Schnirelmann Principle applied on C^1- Banach manifolds.

Structure of the Thesis

The thesis begins with a chapter of Preliminaries presenting the classical notions and results used in proving the results of the subsequent chapters, grouped into two sections: Lebesgue spaces and Sobolev spaces, Definitions, Properties and Some Results from Variational Calculus are included.

The next chapter (Chapter 1 in this summary) is devoted to the study of an eigenvalue problem for the (p, q)-Laplacian operator, in the presence of potentials with nonnegative weights of order p and q, both in the equation and in the boundary conditions. The problem analyzed, denoted above by (P_1), generalizes the Neumann and Steklov cases through the appearance of the spectral parameter λ both in the domain Ω and on its boundary. After formulating the problem and stating the main hypotheses, some auxiliary results and the energy functional $\mathcal{J}_\lambda$ corresponding to the studied problem are introduced. Its properties (differentiability, coercivity, semicontinuity), which are essential for subsequent proofs, are investigated. The analysis branches depending on the position of the exponent r relative to p and q. For $r \in \{p, q\}$, a complete characterization of the spectrum is obtained, being of the form (d, ∞), where $d > 0$ depends on p or q. In the case $r = q$, the approach uses the Nehari manifold and the Lagrange Multipliers Rule. For the cases $r < p$ and $r \in (q, q_*)$, the spectrum is $(0, \infty)$, and the proofs rely on the coercivity of the functional and classical variational techniques. The most delicate case, $r \in (p,q)$ with $r < p_*$, is treated via the fibering method, leading to the determination of an interval of eigenvalues of the form $[\lambda^*, \infty)$, with $\lambda^* > 0$, while for $\lambda < \lambda_* < \lambda^*$ the problem admits no nontrivial solutions. Here, for $\theta \in \{p,q\}$, we have denoted $\theta_* = (N-1)\theta/(N-\theta)$ for $\theta < N$ and $\theta_* = \infty$ for $\theta \geq N$, the critical exponents that follow.

The results presented in this chapter were obtained within the work of Barbu, Burlacu and Moroșanu [2].

The following chapter contains the study of an asymmetric version of the previously investigated problem, denoted above by (P_2), in which only potentials of order q appear, thus removing the symmetry with respect to the exponents p and q. This asymmetric structure leads to a more detailed investigation of the spectral behavior depending on the positioning of the exponents p, q, and r, requiring the treatment of ten distinct cases. After presenting the problem, hypotheses, and main theorems, an auxiliary problem is introduced whose first eigenvalue is essential in proving next results. In eight of the cases, the entire set of eigenvalues of the problem is determined. These are approached via direct methods or by using the Nehari manifold method. For the case when $q < p$ and $r \in (q,p)$, a min-max method based on Krasnosel'skiĭ genus and the Lusternik–Schnirelmann Principle is used. This yields the existence of a sequence of eigenvalues tending to infinity, without being able to conclude that this sequence describes the entire spectrum of the problem. At the same time, this technique allows the extension of some recent results from the literature (especially [27]) regarding the existence of eigenvalues larger than a constant λ_0.

The results of this chapter were published in the article by Barbu, Burlacu and Moroșanu [4].

Chapter 3 is dedicated to the study of the transmission problem (P_3), in which the p- and q-Laplacian operators act in disjoint subdomains of a domain Ω, connected by continuity conditions on the solution and flux equilibrium. After the precise formulation of the problem and stating the hypotheses, the functional space $\widetilde{W}$ is introduced, equivalent to the space of functions with components in $W^{1,p}(\Omega_1)$ and $W^{1,q}(\Omega_2)$, with equal traces on the interface Γ. A family of C^1-class submanifolds $\mathcal{M}_\rho$, $\rho > 0$, each having infinite genus, is introduced in $\widetilde{W}$, making them suitable for the application of the Lusternik–Schnirelmann Principle. Then, an energy functional associated with the problem, denoted $\mathcal{J}$, is defined, and it is shown that its critical points conditioned by the submanifolds introduced before correspond to weak solutions of the problem. It is proved that the functional $\mathcal{J}$ is coercive on these submanifolds and satisfies the Palais–Smale condition, which allows obtaining a sequence of eigenpairs of the problem (P_3) tending to infinity. This result is presented in Theorem 3.1.1. The chapter is structured into three sections, the last two: Section 2.2 is dedicated to preliminary results (including the proof of the properties regarding the infinite genus of the sets $\mathcal{M}_\rho$, $\rho > 0$, and the coercivity of $\mathcal{J}$ on them) and Section 2.3 contains the proof of the main result.

This chapter is based on the work of Barbu, Burlacu and Moroșanu [3].

The thesis concludes with a short chapter containing possible research directions and dissemination of the results.

Keywords: Eigenvalues, (p,q)-Laplacian, Nehari manifold, C^1-manifold, variational methods, nonlinear eigenvalue problem, Krasnosel'skiĭ genus, nonlinear transmission problem, Lusternik–Schnirelmann Principle, Sobolev spaces.

Chapter 1

On an Eigenvalue Problem for the (p, q)-Laplacian with Potentials of Order p and q

In this chapter, we present the original results obtained in collaboration with L. Barbu and G. Moroșanu published in *An. Șt. Univ. Ovidius Constanța* [2].

Among the most important results, we mention: Theorems 1.1.1 and 1.1.2, Lemmas 1.2.1–1.2.3, 1.3.1–1.3.5 and 1.4.1–1.4.8.

To simplify the notation, we will omit the symbols dx and $d\sigma$ in integrals, when the context is clear and there is no ambiguity.

1.1 Formulation of the Problem and Presentation of the Main Results

In this section, we recall the problem formulated in the Introduction, establish the notations used and state the main results of the chapter.

Let $\Omega \subset \mathbb{R}^N$, $N \geq 2$, be a bounded domain with smooth boundary $\partial\Omega$. We consider the eigenvalue problem associated with the operator $\mathcal{A}u = -(\Delta_p u + \Delta_q u)$

$$\begin{cases} \mathcal{A}u + \rho_1(x)|u|^{p-2}u + \rho_2(x)|u|^{q-2}u = \lambda\alpha(x)|u|^{r-2}u, & x \in \Omega, \\ \frac{\partial u}{\partial \nu_{pq}} + \gamma_1(x)|u|^{p-2}u + \gamma_2(x)|u|^{q-2}u = \lambda\beta(x)|u|^{r-2}u, & x \in \partial\Omega. \end{cases} \tag{1.1}$$

In this chapter, we assume the following hypotheses hold

(h_{pqr}) $p,\ q,\ r \in (1, \infty),\ p < q;$

$(h_{\alpha\beta})$ $\alpha \in L^\infty(\Omega)$ and $\beta \in L^\infty(\partial\Omega)$ are nonnegative functions that satisfy

$$\int_\Omega \alpha\ dx + \int_{\partial\Omega} \beta\ d\sigma > 0; \tag{1.2}$$

$(h_{\rho_i\gamma_i})$ $\rho_i \in L^\infty(\Omega)$ and $\gamma_i \in L^\infty(\partial\Omega)$, $i = 1, 2$, are nonnegative functions that satisfy

$$\int_\Omega \rho_i \, dx + \int_{\partial\Omega} \gamma_i \, d\sigma > 0, \; i = 1, 2. \tag{1.3}$$

The solutions u of problem (1.1) belong to the space $W := W^{1,q}(\Omega)$ (since $q > p$) and satisfy equation $(1.1)_1$ in the sense of distributions, while the boundary condition $(1.1)_2$ is satisfied in the sense of the trace.

Definition 1.1.1. *A real number λ is called an eigenvalue of problem (1.1) if there exists $u_\lambda \in W \setminus \{0\}$ such that for every $u \in W$, the following equality holds:*

$$\begin{aligned} &\int_\Omega \left(\mid \nabla u_\lambda \mid^{p-2} + \mid \nabla u_\lambda \mid^{q-2} \right) \nabla u_\lambda \cdot \nabla u + \int_\Omega \left(\rho_1 \mid u_\lambda \mid^{p-2} + \rho_2 \mid u_\lambda \mid^{q-2} \right) u_\lambda u \\ &+ \int_{\partial\Omega} \left(\gamma_1 \mid u_\lambda \mid^{p-2} + \gamma_2 \mid u_\lambda \mid^{q-2} \right) u_\lambda u = \lambda \Big(\alpha \int_\Omega \mid u_\lambda \mid^{r-2} u_\lambda u + \int_{\partial\Omega} \beta \mid u_\lambda \mid^{r-2} u_\lambda u \Big). \end{aligned} \tag{1.4}$$

The function u_λ is called an eigenfunction corresponding to the eigenvalue λ, associated with problem (1.1).

We introduce the following notations:

$$\begin{aligned} K_p(u) &:= \int_\Omega \left(|\nabla u|^p + \rho_1 |u|^p \right) + \int_{\partial\Omega} \gamma_1 |u|^p \, , \\ K_q(u) &:= \int_\Omega \left(|\nabla u|^q + \rho_2 |u|^q \right) + \int_{\partial\Omega} \gamma_2 |u|^q \, , \\ k_\theta(u) &:= \int_\Omega \alpha |u|^\theta + \int_{\partial\Omega} \beta |u|^\theta \quad \forall \, u \in W, \; \theta \in \{p, q, r\}, \end{aligned} \tag{1.5}$$

$$\widehat{\lambda}_q := \inf_{u \in W \setminus \mathcal{Z}} \frac{K_q(u)}{k_q(u)}, \; \widehat{\lambda}_p := \inf_{u \in W \setminus \mathcal{Z}} \frac{K_p(u)}{k_p(u)}, \tag{1.6}$$

$$\begin{aligned} &\lambda_* := \inf_{u \in W \setminus \mathcal{Z}} \Gamma \frac{K_p(u)^\omega K_q(u)^{1-\omega}}{k_r(u)}, \; \lambda^* = \frac{r}{p^\omega q^{1-\omega}} \lambda_*, \\ &\omega := \frac{q-r}{q-p}, \quad \Gamma := \frac{q-p}{(r-p)^{1-\omega}(q-r)^\omega}. \end{aligned} \tag{1.7}$$

Let us note that all eigenfunctions u_λ corresponding to an eigenvalue $\lambda > 0$ satisfy the condition $k_r(u_\lambda) > 0$, thus all eigenfunctions corresponding to problem (1.1) will belong to the set $W \setminus \mathcal{Z}$, where

$$\mathcal{Z} := \{u \in W; \; k_r(u) = 0\}.$$

The main results of this chapter are the following two theorems.

Theorem 1.1.1 ([2])**.** *Assume that the hypotheses (h_{pqr}), $(h_{\alpha\beta})$, $(h_{\rho_i\gamma_i})$ are satisfied.*
(a) If $r = p$, then $\widehat{\lambda}_p > 0$ and the set of eigenvalues of problem (1.1) is the interval $(\widehat{\lambda}_p, \infty)$;
(b) If $r = q$, then $\widehat{\lambda}_q > 0$ and the set of eigenvalues of problem (1.1) is the interval $(\widehat{\lambda}_q, \infty)$.

Theorem 1.1.2 ([2]). *Assume that the hypotheses* $(h_{pqr}), (h_{\alpha\beta})$, *and* $(h_{\rho_i\gamma_i})$ *are satisfied.*
(a) If $\big(r < p\big)$ *or* $\big(r > q$ *with* $r < q(N-1)/(N-q) = q_*$ *in the case* $q < N\big)$, *then the set of eigenvalues of problem* (1.1) *is the interval* $(0, \infty)$;
(b) If $p < r < q$ *with* $r < p(N-1)/(N-p) = p_*$ *in the case* $p < N$, *then* $0 < \lambda_* < \lambda^*$ *and every* $\lambda \in [\lambda^*, \infty)$ *is an eigenvalue of problem* (1.1).

Moreover, for any $\lambda \in (-\infty, \lambda_*)$, *problem* (1.1) *has only the trivial solution.*

Furthermore, the two constants λ_* *and* λ^* *can be expressed as follows:*

$$\lambda_* = \inf_{u \in W\setminus\mathcal{Z}} \frac{K_p(u) + K_q(u)}{k_r(u)}, \quad \lambda^* = \inf_{u \in W\setminus\mathcal{Z}} \frac{\frac{1}{p}K_p(u) + \frac{1}{q}K_q(u)}{\frac{1}{r}k_r(u)}. \tag{1.8}$$

1.2 Auxiliary Results

This section gathers several technical results that will be used in the proofs of the theorems presented earlier.

Lemma 1.2.1 ([2]). *Assume that hypothesis* $(h_{\alpha\beta})$ *is satisfied. If*

$$\theta,\ \tilde{r} \in (1, \infty) \ and \left[\tilde{r} < \theta_* \ if\ \theta < N\right],$$

then

$$\| u \|_{\theta,\tilde{r}} := \| \nabla u \|_{L^\theta(\Omega)} + \big(k_{\tilde{r}}(u)\big)^{\frac{1}{\tilde{r}}} \ \forall\ u \in W^{1,\theta}(\Omega)$$

is a norm on $W^{1,\theta}(\Omega)$, *equivalent to the standard one.*

In the following, for $\theta > 1$, we consider the following eigenvalue problem:

$$\begin{cases} -\Delta_\theta u + \rho(x)|u|^{\theta-2}u = \lambda\alpha(x)|u|^{\theta-2}u \ \text{ in } \Omega, \\ |\nabla u|^{\theta-2}\frac{\partial u}{\partial \nu} + \gamma(x)|u|^{\theta-2}u = \lambda\beta(x)|u|^{\theta-2}u \text{ on } \partial\Omega, \end{cases} \tag{1.9}$$

where $\rho \in L^\infty(\Omega)$ and $\gamma \in L^\infty(\partial\Omega)$ are given nonnegative functions that satisfy

$$\int_\Omega \rho + \int_{\partial\Omega} \gamma \quad > 0. \tag{1.10}$$

We define the following C^1-class functional:

$$\Theta_\theta : W^{1,\theta}(\Omega) \setminus \mathcal{Z}_\theta \to (0, \infty), \quad \Theta_\theta(u) := \frac{K_\theta(u)}{k_\theta(u)} \quad \forall\ u \in W^{1,\theta}(\Omega) \setminus \mathcal{Z}_\theta,$$

where

$$K_\theta(u) := \int_\Omega \big(|\nabla u|^\theta + \rho|u|^\theta\big) + \int_{\partial\Omega} \gamma|u|^\theta.$$

Lemma 1.2.2 ([2]). *Assume that hypothesis* $(h_{\alpha\beta})$ *is satisfied, and that* $\rho \in L^\infty(\Omega)$, $\gamma \in L^\infty(\partial\Omega)$ *are nonnegative functions that satisfy* (1.10). *Then, there exists* $u_* \in W^{1,\theta}(\Omega) \setminus \mathcal{Z}_\theta$ *such that*

$$\Theta_\theta(u_*) = \lambda_\theta := \inf_{u \in W^{1,\theta}(\Omega)\setminus\mathcal{Z}_\theta} \Theta_\theta(u) > 0.$$

Moreover, λ_θ *is the smallest eigenvalue of problem* (1.9), *and* u_* *is an eigenfunction corresponding to the eigenvalue* λ_θ.

For $\lambda > 0$, we define the energy functional associated with problem (1.1):

$$\mathcal{J}_\lambda : W \to \mathbb{R}, \quad \mathcal{J}_\lambda(u) = \frac{1}{p}K_p(u) + \frac{1}{q}K_q(u) - \frac{\lambda}{r}k_r(u), \quad \forall\, u \in W. \tag{1.11}$$

The coercivity of the functional $\mathcal{J}_\lambda$ on W is studied in the following lemma.

Lemma 1.2.3 ([2]). *Assume that the hypotheses (h_{pqr}), $(h_{\alpha\beta})$, $(h_{\rho_i\gamma_i})$ are satisfied. Then, for any $r \in (1, q)$, the functional $\mathcal{J}_\lambda$ is coercive on W, i.e.,*

$$\lim_{\|u\| \to \infty} \mathcal{J}_\lambda(u) = \infty.$$

1.3 Proof of Theorem 1.1.1

In this section, we assume that the hypotheses (h_{pqr}), $(h_{\alpha\beta})$ and $(h_{\rho_i\gamma_i})$ are satisfied and will use them without further mention in the intermediate results.

1.3.1 Proof of Theorem 1.1.1 (a) (Case $r = p$)

The proof of Theorem 1.1.1 (a) is based on the following two lemmas.

Lemma 1.3.1 ([2]). *If $r = p$, then $\widehat{\lambda}_p > 0$ and there are no eigenvalues of problem (1.1) in the interval $(-\infty, \widehat{\lambda}_p]$. Moreover, the following inequality holds:*

$$\widetilde{\lambda}_p := \inf_{u \in W \setminus \mathcal{Z}} \frac{\frac{1}{q}K_q(u) + \frac{1}{p}K_p(u)}{\frac{1}{p}k_p(u)} = \widehat{\lambda}_p. \tag{1.12}$$

Lemma 1.3.2 ([2]). *If $r = p$, then every $\lambda > \widehat{\lambda}_p$ is an eigenvalue of problem (1.1).*

Using Lemmas 1.3.1 and 1.3.2, Theorem 1.1.1 (a) is completely proven.

1.3.2 Proof of Theorem 1.1.1 (b) (Case $r = q$)

We observe that, in the case $r = q$, the functional $\mathcal{J}_\lambda$ takes the following form:

$$\mathcal{J}_\lambda : W \to \mathbb{R}, \quad \mathcal{J}_\lambda(u) = \frac{1}{p}K_p(u) + \frac{1}{q}K_q(u) - \frac{\lambda}{q}k_q(u) \quad \forall\, u \in W. \tag{1.13}$$

In this case, the functional $\mathcal{J}_\lambda$ is no longer coercive on W, so a different method must be used. To this purpose, for $\lambda > 0$, we define the Nehari manifold:

$$\mathcal{N}_\lambda = \left\{u \in W \setminus \{0\};\ \langle \mathcal{J}_\lambda'(u), u\rangle = 0\right\} = \left\{u \in W \setminus \{0\};\ K_p(u) + K_q(u) - \lambda k_q(u) = 0\right\}.$$

Lemma 1.3.3 ([2]). *If $r = q$, then $\widehat{\lambda}_q > 0$ and there are no eigenvalues of problem (1.1) in the interval $(-\infty, \widehat{\lambda}_q]$. Moreover, the following equality holds:*

$$\widetilde{\lambda}_q := \inf_{u \in W \setminus \mathcal{Z}} \frac{\frac{q}{p}K_p(u) + K_q(u)}{k_q(u)} = \widehat{\lambda}_q. \tag{1.14}$$

Lemma 1.3.4 ([2]). *Let $\lambda > \widehat{\lambda}_q$. If $r = q$, then there exists a point $u_* \in \mathcal{N}_\lambda$ at which $\mathcal{J}_\lambda$ attains its minimum over the Nehari manifold $\mathcal{N}_\lambda$, and*

$$m_\lambda := \inf_{u \in \mathcal{N}_\lambda} \mathcal{J}_\lambda(u) > 0.$$

Lemma 1.3.5 ([2]). *Let $\lambda > \widehat{\lambda}_q$. If $r = q$ then the minimizer $u_* \in \mathcal{N}_\lambda$ from Lemma 1.3.4 is an eigenfunction of problem (1.1) corresponding to the eigenvalue λ.*

Using Lemmas 1.3.3, 1.3.4 and 1.3.5, Theorem 1.1.1 (b) is fully proven.

1.4 Proof of Theorem 1.1.2

The proof of Theorem 1.1.2 is based on the following lemmas, in which we assume the hypotheses (h_{pqr}), $(h_{\rho_i\gamma_i})$ and $(h_{\alpha\beta})$ without stating them in each lemma.

1.4.1 Proof of Theorem 1.1.2 (a) (Case $r < p$)

Lemma 1.4.1 ([2]). *If $r < p$ then any $\lambda > 0$ is an eigenvalue of problem (1.1).*

1.4.2 Proof of Theorem 1.1.2 (a) (Case $r \in (q, q_*)$)

Let $\lambda > 0$ be a fixed real number. Since in the case $r \in (q, q_*)$ the functional $\mathcal{J}_\lambda$ is no longer coercive on W, we define a new Nehari manifold as follows:

$$\widehat{\mathcal{N}}_\lambda = \left\{u \in W \setminus \{0\};\ \langle \mathcal{J}'_\lambda(u), u\rangle = K_p(u) + K_q(u) - \lambda k_r(u) = 0\right\}. \tag{1.15}$$

Lemma 1.4.2 ([2]). *Assume that the conditions $q < r$ and $(r < q(N-1)/(N-q) = q_*$ if $q < N)$ are satisfied. Then there exists a point $u_* \in \widehat{\mathcal{N}}_\lambda$ at which the functional $\mathcal{J}_\lambda$ attains its minimum over the manifold $\widehat{\mathcal{N}}_\lambda$ and*

$$m_\lambda := \inf_{u \in \widehat{\mathcal{N}}_\lambda} \mathcal{J}_\lambda(u) > 0.$$

Lemma 1.4.3 ([2]). *Assume that $q < r$ and $(r < q_*$ if $q < N)$. Then the minimizer $u_* \in \widehat{\mathcal{N}}_\lambda$ from Lemma 1.4.2 is an eigenfunction of problem (1.1) corresponding to the eigenvalue λ.*

In conclusion, using Lemmas 1.4.1 through 1.4.3, Theorem 1.1.2 (a) is fully proven.

1.4.3 Proof of Theorem 1.1.2 (b) (Case $r \in (p, q)$)

The proof of this result requires a different approach compared to the previous cases, since the functional $\mathcal{J}_\lambda$ is neither coercive on W nor bounded below on the Nehari manifold.

We will prove Theorem 1.1.2 (b), as in previous cases, using a series of lemmas that assume the hypotheses (h_{pqr}), $(h_{\rho_i\gamma_i})$ and $(h_{\alpha\beta})$ without stating them in each lemma.

Lemma 1.4.4 ([2]). *Assume that $p < r < q$ and $r < p_*$ if $p < N$. Then $0 < \lambda_* < \lambda^*$.*

Lemma 1.4.5 ([2]). *Assume that $p < r < q$ and $r < p_*$ if $p < N$. Then, the constants λ_* and λ^* defined in relation (1.7) can be equivalently expressed as:*

$$\lambda_* = \inf_{u \in W \setminus \mathcal{Z}} \frac{K_p(u) + K_q(u)}{k_r(u)}, \quad \lambda^* = \inf_{u \in W \setminus \mathcal{Z}} \frac{\frac{1}{p}K_p(u) + \frac{1}{q}K_q(u)}{\frac{1}{r}k_r(u)}. \tag{1.16}$$

We define the functional

$$\Phi : W \setminus \mathcal{Z} \to (0, \infty), \quad \Phi(u) := \Gamma \frac{K_p(u)^{\omega} K_q(u)^{1-\omega}}{k_r(u)} \quad \forall u \in W \setminus \mathcal{Z}.$$

Lemma 1.4.6 ([2])**.** *We assume that $p < r < q$ and $r < p_*$ if $p < N$. Then there exists $u_* \in W \setminus \mathcal{Z}$ such that*

$$\lambda_* = \Phi(u_*) = \inf_{u \in W \setminus \mathcal{Z}} \Phi(u).$$

Lemma 1.4.7 ([2])**.** *We assume that $p < r < q$ and $r < p_*$ if $p < N$. If $u_* \in W \setminus \mathcal{Z}$ is the minimizer found in Lemma 1.4.6, then*

$$u^* = \left(\frac{q}{p}\right)^{\frac{1}{q-p}} t(u_*) u_* \in W \setminus \mathcal{Z}, \tag{1.17}$$

where $t(u_*)$ is an eigenfunction of problem (1.1) corresponding to the eigenvalue λ^*.

Moreover,

$$\mathcal{J}_{\lambda^*}(u^*) = 0.$$

Lemma 1.4.8 ([2])**.** *We assume that $p < r < q$ and $r < p_*$ if $p < N$. Then any number $\lambda \in (\lambda^*, \infty)$ is an eigenvalue of problem (1.1) and for each $\lambda \in (-\infty, \lambda_*) \setminus \{0\}$, problem (1.1) has only the trivial solution.*

In conclusion, using Lemmas 1.4.4, 1.4.5, 1.4.7 and 1.4.8, the proof of Theorem 1.1.2 (b) is complete.

Chapter 2

On an Eigenvalue Problem for the (p, q)-Laplacian with q-Type Potentials

This chapter is dedicated to presenting the results obtained in collaboration with L. Barbu and G. Moroșanu published in *Mediterr. J. Math.* [4].
We list the most important among them: Theorems 2.1.1–2.1.3, Lemma 2.2.1, Lemmas 2.3.1–2.3.6, as well as Lemmas 2.4.1–2.4.2.

For the sake of simplicity, in what follows we will omit the measure elements dx and $d\sigma$ from integrals where there is no risk of confusion.

2.1 Problem Formulation and Main Results

In this section we recall the problem formulated in the Introduction, establish the notations used hereafter and state the main results of the chapter.

Let $\Omega \subset \mathbb{R}^N$, with $N \geq 2$, be a bounded domain with smooth boundary $\partial\Omega$. We consider in Ω the eigenvalue problem associated to the operator

$$-\Delta_p - \Delta_q$$

$$\begin{cases} -(\Delta_p u + \Delta_q u) + \rho(x)|u|^{q-2}u = \lambda\alpha(x)|u|^{r-2}u & \text{in } \Omega, \\ \frac{\partial u}{\partial \nu_{pq}} + \gamma(x)|u|^{q-2}u = \lambda\beta(x)|u|^{r-2}u & \text{on } \partial\Omega. \end{cases} \tag{2.1}$$

The entire analysis in this chapter relies on the following hypotheses:

(h_{pqr}) $\quad p, q, r \in (1, \infty), \quad p \neq q$;

$(h_{\alpha\beta})$ $\quad \alpha \in L^\infty(\Omega)$ and $\beta \in L^\infty(\partial\Omega)$ are given nonnegative functions satisfying

$$\int_\Omega \alpha + \int_{\partial\Omega} \beta > 0; \tag{2.2}$$

$(h_{\rho\gamma})$ $\quad \rho \in L^\infty(\Omega)$ and $\gamma \in L^\infty(\partial\Omega)$ are given nonnegative functions such that

$$\int_\Omega \rho + \int_{\partial\Omega} \gamma > 0; \tag{2.3}$$

(h) If $r = q$, then there does not exist a positive constant k_0 such that

$$\rho = k_0\,\alpha \quad \text{a.e. in } \Omega \quad \text{and} \quad \gamma = k_0\,\beta \quad \text{a.e. on } \partial\Omega.$$

Imposing the above hypotheses and analyzing the position of r relative to p and q, we can fully describe the spectrum of the problem stated above in eight out of the ten possible cases. In the other two cases, we obtain only subsets of the spectrum (see Theorems 2.1.1–2.1.3 presented below).

Since we only assumed $p \neq q$, the solution u of problem (2.1) is an element of the Sobolev space $W := W^{1,\max\{p,q\}}(\Omega)$, which satisfies equation $(2.1)_1$ in the distributional sense and the boundary condition $(2.1)_2$ in the trace sense. In this sense, we have the following definition.

Definition 2.1.1. *(i) A function $u \in W$ is called a* weak solution *of problem* (2.1) *if*

$$\begin{aligned} \int_\Omega (|\nabla u|^{p-2} + |\nabla u|^{q-2})\nabla u \cdot \nabla v + \int_\Omega \rho |u|^{q-2}uv + \int_{\partial\Omega} \gamma |u|^{q-2}uv \\ = \lambda\Big(\int_\Omega \alpha |u|^{r-2}uv + \int_{\partial\Omega} \beta |u|^{r-2}uv\Big) \quad \forall v \in W. \end{aligned} \tag{2.4}$$

(ii) A real number λ is called an eigenvalue *of problem* (2.1) *if it admits a nontrivial weak solution $u_\lambda \in W \setminus \{0\}$. In this case, the function u_λ is called an* eigenfunction *corresponding to the eigenvalue λ and the pair (λ, u_λ) is called an* eigenpair *of problem* (2.1).

All eigenfunctions of problem (2.1) satisfy the relation

$$j_r(u_\lambda) > 0,$$

where

$$j_r(u) := \int_\Omega \alpha |u|^r + \int_{\partial\Omega} \beta |u|^r \quad \forall u \in W, \tag{2.5}$$

hence they belong to the set

$$W \setminus \mathcal{Z}, \quad \mathcal{Z} := \{u \in W : j_r(u) = 0\}. \tag{2.6}$$

We also introduce the following constants, which will play an important role in the results obtained below:

$$\Lambda_q := \inf_{u \in W \setminus \mathcal{Z}} \frac{\int_\Omega (|\nabla u|^q + \rho |u|^q) + \int_{\partial\Omega} \gamma |u|^q}{j_r(u)}, \quad \lambda_0 := \frac{\int_\Omega \rho + \int_{\partial\Omega} \gamma}{j_r(1)}. \tag{2.7}$$

Moreover, for any $\sigma > 0$, we define the set

$$\mathcal{M}_\sigma := \{u \in W : j_r(u) = \sigma\}. \tag{2.8}$$

The main results of the chapter are the following three theorems.

Theorem 2.1.1 ([4])**.** *Assume that the hypotheses (h_{pqr}), $(h_{\alpha\beta})$ and $(h_{\rho\gamma})$ are satisfied. If $r = q$ and the assumption (h) holds, then the constants Λ_q and λ_0 defined in* (2.7) *are positive. Moreover, $\Lambda_q < \lambda_0$ and any $\lambda \in (\Lambda_q, \lambda_0)$ is an eigenvalue of problem* (2.1)*. Furthermore, problem* (2.1) *has only trivial solutions for λ in the interval $(-\infty, \Lambda_q]$.*

Theorem 2.1.2 ([4])**.** *Assume that the hypotheses $(h_{pqr}), (h_{\alpha\beta})$ and $(h_{\rho\gamma})$ are satisfied. In each of the following cases*

(a) $r = p$; (b) $\max\{p,q\} < r < \max\{p_*, q_*\}$; (c) $r < \min\{p,q\}$; (d) $p < r < q$, *the set of eigenvalues of problem (2.1) is the interval* $(0,\infty)$*.*

Theorem 2.1.3 ([4])**.** *Assume that the hypotheses* (h_{pqr}), $(h_{\alpha\beta})$ *and* $(h_{\rho\gamma})$ *are satisfied. If* $r = q$ *or* $q < r < p$*, then for every* $\sigma > 0$ *the problem (2.1) has infinitely many pairs of eigen-solutions of the form*

$$(\lambda_n, \pm u_n) \in \mathbb{R} \times \mathcal{M}_\sigma \quad \text{with} \quad \lambda_n \to \infty \quad \text{as} \quad n \to \infty.$$

2.2 Auxiliary Results

For clarity we use the following notations:

$$K_\xi(u) := \int_\Omega |\nabla u|^\xi, \quad \xi \in \{p,q\}, \quad k_q(u) := \int_\Omega \rho|u|^q + \int_{\partial\Omega} \gamma|u|^q, \quad u \in W. \tag{2.9}$$

Next, we state an auxiliary result in order to prove the theorems stated earlier.
For $\theta > 1$, consider the following eigenvalue problem

$$\begin{cases} -\Delta_\theta u + \rho(x)|u|^{\theta-2}u = \lambda\alpha(x)|u|^{\theta-2}u & \text{in } \Omega, \\ |\nabla u|^{\theta-2}\frac{\partial u}{\partial \nu} + \gamma(x)|u|^{\theta-2}u = \lambda\beta(x)|u|^{\theta-2}u & \text{on } \partial\Omega. \end{cases} \tag{2.10}$$

We define the C^1-functional

$$\Theta : W^{1,\theta}(\Omega) \setminus \mathcal{Z}_\theta \to (0,\infty), \quad \Theta(u) := \frac{K_\theta(u) + k_\theta(u)}{j_\theta(u)} \quad \forall u \in W^{1,\theta}(\Omega) \setminus \mathcal{Z}_\theta.$$

The following lemma provides an important characterization of the minimum value of the functional Θ on the set $W^{1,\theta}(\Omega) \setminus \mathcal{Z}_\theta$.

Lemma 2.2.1 ([4])**.** *There exists* $u_* \in W^{1,\theta}(\Omega) \setminus \mathcal{Z}_\theta$ *such that*

$$\Theta(u_*) = \lambda_\theta := \inf_{u \in W^{1,\theta}(\Omega)\setminus\mathcal{Z}_\theta} \Theta(u) > 0. \tag{2.11}$$

Moreover, λ_θ *is the smallest eigenvalue of problem (2.10) and* u_* *is an eigenfunction corresponding to the eigenvalue* λ_θ*.*

2.3 Proof of Theorems 2.1.1 and 2.1.2

The proof of the results in this section will be carried out with the help of intermediate lemmas, where we assume that the hypotheses (h_{pqr}), $(h_{\alpha\beta})$ and $(h_{\rho\gamma})$ are satisfied. Moreover, in the particular case $r = q$, we will also assume that hypothesis (h) holds.

Lemma 2.3.1 ([4])**.** *If* $r = q$*, then* $\Lambda_q > 0$ *and there are no eigenvalues of problem (2.1) in the interval* $(-\infty, \Lambda_q]$*. Moreover,*

$$\Lambda_q = \widetilde{\Lambda}_q := \inf_{u \in W\setminus\mathcal{Z}} \frac{\frac{1}{q}\big(K_q(u) + k_q(u)\big) + \frac{1}{p}K_p(u)}{\frac{1}{q}j_q(u)}. \tag{2.12}$$

The following lemma expresses an inequality relation between the variational constants defined in (2.7).

Lemma 2.3.2 ([4])**.** *If* $r = q$*, then* $\Lambda_q < \lambda_0$*.*

Let $\lambda > 0$. Consider the functional $\mathcal{J}_{r\lambda} : W \to \mathbb{R}$, of class C^1, defined by

$$\mathcal{J}_{r\lambda}(u) = \frac{1}{p}K_p(u) + \frac{1}{q}\big(K_q(u) + k_q(u)\big) - \frac{\lambda}{r}j_r(u). \tag{2.13}$$

Regarding the coercivity of the functional $\mathcal{J}_{r\lambda}$, we have the following result.

Lemma 2.3.3 ([4]). *Assume that one of the following two conditions is satisfied:*

(i) $1 < r < q$ *and* $\lambda > 0$*;*

(ii) $r = q < p$ *and* $\lambda \in (\Lambda_q, \lambda_0)$.

Then the functional $\mathcal{J}_{r\lambda}$ *is coercive on the space* W*, i.e.,*

$$\lim_{\|u\|\to\infty} \mathcal{J}_{r\lambda}(u) = \infty.$$

Assuming the conditions of Lemma 2.3.3 hold, we obtain the following result regarding the existence of eigenvalues for problem (2.1).

Lemma 2.3.4 ([4]). *(i) If* $r < q$*, then every real number* $\lambda > 0$ *is an eigenvalue of problem* (2.1)*;*

(ii) If $r = q < p$*, then every real number* $\lambda \in (\Lambda_q, \lambda_0)$ *is an eigenvalue of problem* (2.1).

Next, we present the complementary cases to those considered in Lemma 2.3.4.

If $r > q$ or $r = q > p$, we cannot expect the functional $\mathcal{J}_{r\lambda}$ to remain coercive on W. Therefore, for $\lambda > 0$, we consider the Nehari manifold associated with it, defined as

$$\mathcal{N}_{r\lambda} = \{u \in W \setminus \{0\} : \langle \mathcal{J}'_{r\lambda}(u), u\rangle = K_p(u) + K_q(u) + k_q(u) - \lambda j_r(u) = 0\}.$$

Lemma 2.3.5 ([4]). *Assume that one of the following two hypotheses is fulfilled:*

(i) $r = q > p$ *and* $\lambda \in (\Lambda_q, \lambda_0)$*;*

(ii) $r \neq q$, $\max\{p, q\} \le r < \max\{p_*, q_*\}$ *and* $\lambda > 0$*, where* p_* *and* q_* *are the critical exponents defined later.*

Then there exists $u_* \in \mathcal{N}_{r\lambda}$ *at which the functional* $\mathcal{J}_{r\lambda}$ *attains its minimum on the manifold* $\mathcal{N}_{r\lambda}$*, i.e.,*

$$m_{r\lambda} := \inf_{u\in\mathcal{N}_{r\lambda}} \mathcal{J}_{r\lambda}(u) > 0.$$

The last result necessary for the proof of Theorems 2.1.1 and 2.1.2 shows that the minimum $u_* \in \mathcal{N}_{r\lambda}$ obtained in Lemma 2.3.5 is, in fact, a critical point of the functional $\mathcal{J}_{r\lambda}$.

Lemma 2.3.6 ([4]). *(i) If* $r = q > p$*, then any real number* $\lambda \in (\Lambda_q, \lambda_0)$ *is an eigenvalue of problem* (2.1)*;*

(ii) If $r \neq q$ *and* $\max\{p, q\} \le r < \max\{p_*, q_*\}$*, then any real number* $\lambda \in (0, \infty)$ *is an eigenvalue of problem* (2.1).

In conclusion, using Lemma 2.3.4 and Lemma 2.3.6, Theorems 2.1.1 and 2.1.2 are fully proved.

2.4 Proof of Theorem 2.1.3

We analyze the case $q < r < p$, where the functional $\mathcal{J}_{r\lambda}$ is neither coercive on the space W nor bounded below on the manifold $\mathcal{N}_{r\lambda}$.

If $r = q$ and $\lambda \geq \lambda_0$, as we noted in the previous section, we will need to consider other arguments to deduce that problem (2.1) has eigenvalues $\lambda \geq \lambda_0$.
Consequently, to obtain the multiplicity result stated in Theorem 2.1.3, we will use the notion of genus in the sense of Krasnosel'skiĭ.

One of the most important properties of the manifold $\mathcal{M}_\sigma$ is stated in the following result.

Lemma 2.4.1 ([4])**.** *For every positive integer k, there exists a compact, symmetric set $K \subset \mathcal{M}_\rho$ such that its genus satisfies $\gamma(K) = k$.*

Consider the following functional $\mathcal{J} : W \to \mathbb{R}$, defined by

$$\mathcal{J}(u) = \frac{1}{p} K_p(u) + \frac{1}{q}\big(K_q(u) + k_q(u)\big) \quad \forall u \in W. \tag{2.14}$$

An important property of the functional defined above is given by the following lemma.

Lemma 2.4.2 ([4])**.** *If $r = q$ or $q < r < p$, then the functional $\mathcal{J}$ restricted to the manifold $\mathcal{M}_\sigma$ satisfies the Palais–Smale condition. That is, every sequence $\{u_n\} \subset \mathcal{M}_\sigma$ such that the sequence $\{\mathcal{J}(u_n)\}$ is bounded and*

$$\mathcal{J}'_{\mathcal{M}_\sigma}(u_n) \to 0,$$

contains a convergent subsequence.

Finally, using the Lusternik–Schnirelmann Principle and the lemmas above, one can conclude that there exists an infinite sequence of critical points

$$\pm u_n, \quad n \geq 1,$$

for the functional $\mathcal{J}$ in $\mathcal{M}_\sigma$.

To each critical point $\pm u_n$ correspond Lagrange multipliers λ_n, thus yielding an infinite sequence of eigenpairs of problem (2.1) of the form

$$(\lambda_n, \pm u_n) \in (0, \infty) \times \mathcal{M}_\sigma, \quad n \geq 1.$$

Moreover, the sequence $\lambda_n \to \infty$ as $n \to \infty$, which completes the proof of Theorem 2.1.3.

Chapter 3

A Nonlinear Transmission Eigenvalue Problem with a Neumann–Robin Type Boundary Condition

This chapter gathers a series of results obtained in collaboration with L. Barbu and G. Moroșanu, published in *Math. Methods Appl. Sci.* [3].

Among these, we mention: Theorem 3.1.1, Lemmas 3.2.1, 3.2.2, as well as Lemmas 3.3.1–3.3.3.

For the sake of simplifying notations, we will omit the measures dx and $d\sigma$ in integrals whenever the context is clear and no ambiguity arises.

3.1 Problem Formulation and Presentation of the Main Results

In this section, we recall the problem mentioned in the Introduction, establish the necessary notations and state the main result of the chapter.

Let $\Omega \subset \mathbb{R}^N$, $N \geq 2$ be a bounded domain with smooth boundary Σ, and let Ω_1 be a subdomain with smooth boundary Γ such that $\overline{\Omega}_1 \subset \Omega$ and $\Omega_2 = \Omega \setminus \overline{\Omega}_1$.

We consider in Ω the following transmission eigenvalue problem:

$$\begin{cases} -\Delta_p u_1 + \gamma_1(x)|u_1|^{r-2}u_1 = \lambda |u_1|^{p-2}u_1 & \text{in } \Omega_1, \\ -\Delta_q u_2 + \gamma_2(x)|u_2|^{s-2}u_2 = \lambda |u_2|^{q-2}u_2 & \text{in } \Omega_2, \\ u_1 = u_2, \quad \frac{\partial u_1}{\partial \nu_p} = \frac{\partial u_2}{\partial \nu_q} & \text{on } \Gamma, \\ \frac{\partial u_2}{\partial \nu} + \beta(x)|u_2|^{\zeta-2}u_2 = 0 & \text{on } \Sigma, \end{cases} \tag{3.1}$$

where λ is a real parameter.

Throughout the chapter, we will assume that the following hypotheses hold:

$$(h_1) \quad \begin{cases} p, q, r, s, \zeta \in (1, \infty), \quad p \leq q, \quad \zeta < q_*, \\ r < p\left(1 + \frac{p}{N}\right) \quad \text{if } r > p \text{ and } p < N, \\ s < q\left(1 + \frac{q}{N}\right) \quad \text{if } s > q \text{ and } q < N. \end{cases} \tag{3.2}$$

(h_2) $\quad \gamma_i \in L^\infty(\Omega_i), \quad i = 1, 2, \quad \beta \in L^\infty(\Sigma), \quad \beta \geq 0$ a.e. on Σ.

For $1 < \theta \leq \infty$, we denote the usual norms of the Lebesgue spaces $L^\theta(\Omega_i)$ and $L^\theta(\Sigma)$ by $\|\cdot\|_{i\theta},\ i = 1, 2$, respectively $\|\cdot\|_{\partial\theta}$.

Obviously, the solutions $u = (u_1, u_2)$ of problem (3.1) belong to the space

$$W := \{u \in W^{1,p}(\Omega); \quad u|_{\Omega_2} \in W^{1,q}(\Omega_2)\},$$

where $u_i := u|_{\Omega_i},\ i = 1, 2$. On W we consider the usual norm

$$\|u\| := \|u_1\|_1 + \|u_2\|_2 \quad \forall u = (u_1, u_2) \in W, \tag{3.3}$$

where the norms $\|\cdot\|_i,\ i = 1, 2$ are defined by

$$\|u_1\|_1 := \|\nabla u_1\|_{1p} + \|u_1\|_{1p}, \quad \|u_2\|_2 := \|\nabla u_2\|_{2q} + \|u_2\|_{2q}. \tag{3.4}$$

The space W defined above can be identified with the space

$$\widetilde{W} := \left\{\widetilde{u} = (u_1, u_2) \in W^{1,p}(\Omega_1) \times W^{1,q}(\Omega_2); \quad u_1 = u_2 \text{ on } \Gamma\right\}, \tag{3.5}$$

which implies that W is a reflexive Banach space.

Definition 3.1.1. *A real number λ is called an* eigenvalue *of problem* (3.1) *if the problem admits a weak solution $\widetilde{u}_\lambda = (u_{1\lambda}, u_{2\lambda}) \in \widetilde{W} \setminus \{(0, 0)\}$.*
In this case, $\widetilde{u}_\lambda$ is called an eigenfunction *associated to the eigenvalue λ and the pair $(\lambda, \widetilde{u}_\lambda)$ is called an* eigenpair *of problem* (3.1).

The following result follows by an argument similar to the one used in [9, Proposition 1.1]. It provides a characterization of the eigenvalues of problem (3.1).

Proposition 3.1.1 ([3])**.** *A real number λ is an eigenvalue of problem* (3.1) *if and only if there exists $\widetilde{u}_\lambda = (u_{1\lambda}, u_{2\lambda}) \in \widetilde{W} \setminus \{(0, 0)\}$ such that for every $(v_1, v_2) \in \widetilde{W}$ the following equality holds:*

$$\begin{aligned}
&\int_{\Omega_1} |\nabla u_{1\lambda}|^{p-2}\nabla u_{1\lambda} \cdot \nabla v_1 + \int_{\Omega_2} |\nabla u_{2\lambda}|^{q-2}\nabla u_{2\lambda} \cdot \nabla v_2 \\
&\quad + \int_{\Omega_1} \gamma_1 |u_{1\lambda}|^{r-2} u_{1\lambda} v_1 + \int_{\Omega_2} \gamma_2 |u_{2\lambda}|^{s-2} u_{2\lambda} v_2 \\
&\quad + \int_{\Sigma} \beta |u_{2\lambda}|^{\zeta-2} u_{2\lambda} v_2 \, d\sigma \\
&\quad = \lambda \left(\int_{\Omega_1} |u_{1\lambda}|^{p-2} u_{1\lambda} v_1 + \int_{\Omega_2} |u_{2\lambda}|^{q-2} u_{2\lambda} v_2 \right).
\end{aligned} \tag{3.6}$$

Let $\rho > 0$. Consider the subset $\mathcal{M}_\rho$ of the space $\widetilde{W}$ defined by

$$\mathcal{M}_\rho := \left\{\widetilde{u} = (u_1, u_2) \in \widetilde{W} \ : \ \frac{1}{p}\int_{\Omega_1} |u_1|^p + \frac{1}{q}\int_{\Omega_2} |u_2|^q = \rho\right\}. \tag{3.7}$$

The main result of this chapter is the following theorem.

Theorem 3.1.1. ([3]) *Assume hypotheses (h_1) and (h_2) hold. Then, for every $\rho > 0$, there exists a sequence of eigenpairs*

$$\left(\lambda_n, \pm(u_{1n}, u_{2n})\right)_n$$

of problem (3.1)*, with*

$$\left((u_{1n}, u_{2n})\right)_n \subset \mathcal{M}_\rho$$

and

$$\lambda_n \to +\infty \quad as \quad n \to \infty.$$

3.2 Auxiliary Results

In this section we present some properties of the set $\mathcal{M}_\rho$, defined in (3.7), which are used in the proof of the main result.

We introduce the following notations:

$$\begin{aligned} K_{pq}(u_1,u_2) &:= \frac{1}{p}\int_{\Omega_1} |\nabla u_1|^p + \frac{1}{q}\int_{\Omega_2} |\nabla u_2|^q, \\ k_{rs\zeta}(u_1,u_2) &:= \frac{1}{r}\int_{\Omega_1} \gamma_1|u_1|^r + \frac{1}{s}\int_{\Omega_2} \gamma_2|u_2|^s + \frac{1}{\zeta}\int_{\Sigma} |u_2|^\zeta, \\ j_{pq}(u_1,u_2) &:= \frac{1}{p}\int_{\Omega_1} |u_1|^p + \frac{1}{q}\int_{\Omega_2} |u_2|^q \quad \forall (u_1,u_2) \in \widetilde{W}. \end{aligned} \tag{3.8}$$

We define a C^1-class functional $\mathcal{J} : \widetilde{W} \to \mathbb{R}$ by

$$\mathcal{J}(\widetilde{u}) = K_{pq}(u_1,u_2) + k_{rs\zeta}(u_1,u_2) \quad \forall \widetilde{u} = (u_1,u_2) \in \widetilde{W}. \tag{3.9}$$

Clearly, the function $j_{pq} : \widetilde{W} \to \mathbb{R}$ is of class C^1. Due to the fact that for every $\widetilde{u} = (u_1,u_2) \in \mathcal{M}_\rho$, we have $\langle j'_{pq}(\widetilde{u}), \widetilde{u}\rangle \neq 0$, it follows that ρ is a regular value of this functional. Therefore, $\mathcal{M}_\rho = j_{pq}^{-1}(\rho)$ is a Banach C^1-submanifold of $\widetilde{W}$ with codimension 1.

Moreover, the tangent space at a point $\widetilde{u} = (u_1,u_2) \in \mathcal{M}_\rho$ is given by the equality

$$T_{\widetilde{u}}\mathcal{M}_\rho = \ker j'_{pq}(\widetilde{u}). \tag{3.10}$$

We define the C^1-functional $\mathcal{J} : \widetilde{W} \to \mathbb{R}$ again by

$$\mathcal{J}(\widetilde{u}) = K_{pq}(u_1,u_2) + k_{rs\zeta}(u_1,u_2) \quad \forall \widetilde{u} = (u_1,u_2) \in \widetilde{W}. \tag{3.11}$$

Clearly, $\mathcal{J} \in C^1(\mathcal{M}_\rho, \mathbb{R})$.

We denote by $\mathcal{J}_{\mathcal{M}_\rho}$ the restriction of $\mathcal{J}$ to $\mathcal{M}_\rho$ and by $\mathcal{J}'_{\mathcal{M}_\rho}(\widetilde{u})$ the differential of $\mathcal{J}$ at $\widetilde{u} \in \mathcal{M}_\rho$ relative to $\mathcal{M}_\rho$, i.e., the restriction of $\mathcal{J}'(\widetilde{u})$ to the tangent space $T_{\widetilde{u}}\mathcal{M}_\rho$.

Lemma 3.2.1 ([3])**.** *At any point $\widetilde{u} \in \mathcal{M}_\rho$, the differential of $\mathcal{J}$ relative to $\mathcal{M}_\rho$ satisfies the equality*

$$\mathcal{J}'_{\mathcal{M}_\rho}(\widetilde{u}) = \mathcal{J}'(\widetilde{u}) - \lambda(\widetilde{u}) j'_{pq}(\widetilde{u}), \quad where \quad \lambda(\widetilde{u}) = \frac{\langle \mathcal{J}'(\widetilde{u}), \widetilde{u}\rangle}{\langle j'_{pq}(\widetilde{u}), \widetilde{u}\rangle}. \tag{3.12}$$

The following lemma establishes that $\mathcal{M}_\rho$ has infinite genus.

Lemma 3.2.2 ([3])**.** *For every positive integer k, there exists a symmetric and compact subset $K \subset \mathcal{M}_\rho$ such that $\gamma(K) = k$.*

3.3 Proof of Theorem 3.1.1

In this section, we assume that hypotheses (h_1) and (h_2) are satisfied and we will use them without further mention.

The proof of Theorem 3.1.1 follows as a consequence of some intermediate results.

Lemma 3.3.1 ([3]). *The functional $\mathcal{J}_{\mathcal{M}_\rho}$ is coercive, i.e.,*

$$\lim_{\|(u_1,u_2)\|\to\infty,(u_1,u_2)\in\mathcal{M}_\rho} \mathcal{J}(u_1,u_2) = +\infty.$$

An important role in the proof of the main result is played by the following inequality.

Lemma 3.3.2 ([3]). *Let $\mathcal{K} := K'_{pq} : \widetilde{W} \to \widetilde{W}^*$ be the derivative of the functional K_{pq} defined in (3.8)$_1$. Then, for every $\widetilde{u} = (u_1,u_2), \widetilde{v} = (v_1,v_2) \in \widetilde{W}$, the following inequality holds:*

$$\begin{aligned} \langle \mathcal{K}(\widetilde{u}) - \mathcal{K}(\widetilde{v}), \widetilde{u} - \widetilde{v}\rangle &\geq \big(\|\nabla u_1\|_{1p}^{p-1} - \|\nabla v_1\|_{1p}^{p-1}\big)\big(\|\nabla u_1\|_{1p} - \|\nabla v_1\|_{1p}\big) \\ &\quad + \big(\|\nabla u_2\|_{2q}^{q-1} - \|\nabla v_2\|_{2q}^{q-1}\big)\big(\|\nabla u_2\|_{2q} - \|\nabla v_2\|_{2q}\big) \\ &\geq 0. \end{aligned} \tag{3.13}$$

Lemma 3.3.3 ([3]). *The functional $\mathcal{J}$ satisfies the Palais–Smale condition relative to the manifold $\mathcal{M}_\rho$.*

The existence of an infinite number of critical points $\pm\widetilde{u}_n$, $n \geq 1$, for the functional $\mathcal{J}$ on the set $\mathcal{M}_\rho$ is a consequence of Lemmas 3.2.2, 3.3.1, 3.3.3 and the Lusternik–Schnirelmann theorem.

To each critical point $\pm\widetilde{u}_n$, $n \geq 1$, corresponds a Lagrange multiplier λ_n, which leads to an infinite sequence of eigenpairs $(\lambda_n, \pm\widetilde{u}_n)$, $n \geq 1$, of problem (3.1).

Finally, it is shown that $\lambda_n \to \infty$, thus completing the proof of Theorem 3.1.1.

General Conclusions

Closing Remarks

The results obtained in this thesis contribute to the extension of the spectral theory for nonlinear operators of (p, q)-Laplacian in the presence of potentials of order q and/or p, as well as generalized boundary conditions. The study of the two problems presented in Chapters 1 and 2 allowed the derivation of complete or partial characterizations of the spectrum, through the application of combined variational methods adapted to the considered context.

Moreover, the analysis in Chapter 3 of a nonlinear transmission problem, where the p- and q-Laplacian operators act on different subdomains, addressed a topic less explored in the literature but with important theoretical potential.

Future Research Directions

The results presented in this thesis suggest several possible directions for further research. Among these, we mention:

(i) Obtaining similar results to those in Chapters 1 and 2 in the case where potentials with indefinite weights (i.e., those that may change sign) are introduced. In such cases, some of the arguments used in this work no longer apply, thus requiring the use of alternative techniques;

(ii) The analysis of transmission problems with multiple subdomains and different elliptic operators in each subdomain, including cases where the subdomains interact via nonlinear flux conditions;

(iii) Investigation of eigenvalue problems associated with the operators studied in this work in the context of unbounded domains, for example, on $\mathbb{R}^N$ or on $\mathbb{R}^N \setminus \overline{\Omega}$, where $\Omega \subset \mathbb{R}^N$ is a bounded domain.

Dissemination of Results

The results obtained within this thesis have been disseminated through the following published articles and conference presentations:

Published Articles

L. Barbu, **A. Burlacu**, G. Moroșanu, *An eigenvalue problem involving the (p, q)-Laplacian with a parametric boundary condition*, Mediterr. J. Math. **20**(4), art. no. 232, 2023 (Q1 JIF quartile, Q2 AIS quartile).
https://doi.org/10.1007/s00009-023-02431-0

L. Barbu, **A. Burlacu**, G. Moroșanu, *On a nonlinear transmission eigenvalue problem with a Neumann–Robin boundary condition*, Math. Methods Appl. Sci., **46**(17), 18375–18386, 2023 (Q1 JIF quartile, Q3 AIS quartile).
https://doi.org/10.1002/mma.9563

L. Barbu, **A. Burlacu**, G. Moroșanu, *On an eigenvalue problem associated with the* (p, q)*-Laplacian*, An. Șt. Univ. Ovidius Constanța, **32**(1), 45–63, 2024 (Q2 JIF quartile, Q4 AIS quartile).
https://doi.org/10.2478/auom-2024-0003

Conference Presentations

International Conferences

Eigenvalues of (p,q)-Laplacian under Robin-Steklov Boundary Condition, The 10th International Scientific Conference-Sea Conf, Mircea cel Bătrân Naval Academy, May 16-18, 2024 (oral presentation);
https://www.anmb.ro/ro/conferinte/sea-conf/

Continuous Spectrum for an Eigenvalue Problem Governed by the (p,q)-Laplacian, ICATA-International Conference on Approximation Theory and Its Applications, Lucian Blaga University of Sibiu, July 17-20, 2024 (oral presentation);
https://conferences.ulbsibiu.ro/icata/

On a Nonlinear Transmission Eigenvalue Problem, The 11th International Scientific Conference -Sea Conf, Mircea cel Bătrân Naval Academy, May 15-17, 2025 (oral presentation);
https://www.anmb.ro/ro/conferinte/sea-conf/

Transmission Eigenvalue Problems with Neumann-Robin Boundary Conditions Involving the p- and q-Laplacian, The 6th International Conference on Mathematics and Its Applications in Science and Engineering, University of Plovdiv, July 15-17, 2025 (oral presentation);
https://www.icmase.com/

Analysis of a Generalized Robin-Steklov Eigenvalue Problem with (p,q)-Laplacian, The 6th International Conference on Mathematics and Its Applications in Science and Engineering, University of Plovdiv, July 15-17, 2025 (oral presentation).
https://www.icmase.com/

National Conferences

An Eigenvalue Problem Involving the (p, q)-Laplacian with a Parametric Boundary Condition, Mathematics Communications Session, Faculty of Mathematics and Computer Science, Ovidius University of Constanța, December 10, 2022 (oral presentation);
https://fmi.univ-ovidius.ro/sesiunea-de-comunicari-matematice-2022/

On an Eigenvalue Problem Associated with the (p, q)-Laplacian, Mathematics Communications Session, Faculty of Mathematics and Computer Science, Ovidius University of Constanța, December 9, 2023 (oral presentation).
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