"OVIDIUS" UNIVERSITY FROM CONSTANȚA

DOCTORAL SCHOOL OF MATHEMATICS

MATHEMATICS DOCTORAL DOMAIN

Summary of the Phd Thesis

Linear difference equations of degree n and some of their applications

PhD Coordinator,
Dr.habil.Prof.Univ.
Cristina FLAUT

PhD Student,
Mariana-Geanina TUDOR
(married ZAHARIA)

CONSTANȚA, 2024

Contents

1. Motivation

I wanted to pursue a PhD in mathematics and consider it a valuable choice because as a doctoral student in mathematics, I have been involved in cutting-edge research and it has been a challenge to add new ideas, techniques, and methods to what is already known. I have encountered complex problems and questions and had to develop my own methods to address them. As I became more involved in mathematical research, I have also developed my communication skills, as I have had to communicate my ideas to other specialists in the field. This has been a challenge during research project presentations, reports, mathematical communication sessions, conferences or symposia, or discussions with mathematics colleagues.

I chose this specialization as a continuation of my professional development, given that I am a mathematics teacher at the "Constantin Brătescu" National Pedagogical College in Constanța, with 26 years of experience in the field of education.
During my doctoral training, I presented the following papers at ISI-indexed conferences:

1) I attended the 8th International Eurasian Conference on Mathematical Sciences And Applications Baku / Azerbaijan, presenting the paper "Some remarks on difference linear equations of degree n" from August 27-30, 2019, http://www.iecmsa.org/organize/2019 , http://www.iecmsa.org/2019/, published in Mathematical Methods in the Applied Sciences, DOI: 10.1002/mma.6257, https://onlinelibrary.wiley.com/doi/full/10.1002/mma.6257

2) I participated at International Conference Prishtina, KOSOVO, 2021,presenting the paper "Equations with quaternions" from June 11-12, http://fwbcma2021.ilirias.com/

3) I participated at 10th International Eurasian Conference on Mathematical Sciences and Applications in Sakarya, (IECMSA-2021),TURKEY, presenting the paper "Equations with quaternions and octonions" from August 25-27,2021
http://www.iecmsa.org/program/,
https://us02web.zoom.us/j/2830171452success

4) I attended the 11th International Eurasian Conference on Mathematical Sciences and Applications in Istanbul,(IECMSA-2022) TURKEY, presenting the paper "Quaternions Quadratic Equation" from August 29 to September 1, 2022
http://www.iecmsa.org/
e-mail: conference@iecmsa.org
http://www.iecmsa.org/program/

I have also participated in the following scientific events:

1) IMCS-55 - The Fifth Conference of the Mathematical Society of the Republic of Moldova dedicated to the 55th anniversary of the foundation of the

"Vladimir Andrunachievici" Institute of Mathematics and Computer Science, September 28-October 1, 2019, Chișinău,
where I presented the paper: "Some remarks regarding Fibonacci and Lucas numbers", http://www.math.md/imcs55/index.htm

2) International Symposium Actual Problems Of Mathematics And Informatics Tum, Chisinau, Republic of Moldova, November 27-28, 2020,
"Applications of some special number sequences and quaternion elements using Fibonacci and Lucas elements",
https://utm.md/wp-content/uploads/2020/11/Programme-SIPAMI-2020.pdf

3) Mathematical Communications Session, Constanta, December 5, 2020
https://fmi.univ-ovidius.ro/wp-content/uploads/2020/anunturi/main/scm-sectb.pdf
where I presented the paper: "Recurrence relations. Applications in cryptography", https://fmi.univ-ovidius.ro/sesiunea-de-comunicari-matematice-2020/

4) Mathematical Communications Session, Constanta, December 11, 2021
where I presented the paper: "Solving Equations over Quaternions and Octonions", https://fmi.univ-ovidius.ro/sesiunea-de-comunicari-matematice-2021/

5) I participated in the International Conference RESTART4EDU: Academic Teaching and Learning Opportunities (1st edition), under the code 2020-1-RO01-KA226-HE-095772, funded by the Erasmus Plus Program. The conference took place in Constanța on April 27-28, 2023, where I presented a paper titled "The Romanian undergraduate students' perspective on project-based learning". The paper was published in The "Black Sea" Journal of Psychology, Vol. 14, Issue 1, 45-52, Spring, 2023 ISSN: 2068-4649 www.bspsychology.ro,
https://bspsychology.ro/index.php/BSJoP

As a PhD student at Ovidius University in Constanta, I have received a grant recipient from September 1, 2020 to August 31, 2021, as part of the project "Excelența academică și valori antreprenoriale - sistem de burse pentru asigurarea oportunităților de formare și dezvoltare a competențelor antreprenoriale ale doctoranzilor și postdoctoranzilor" (ANTREPRENORDOC) ("Academic Excellence and Entrepreneurial Values - a scholarship system for providing opportunities to develop the entrepreneurial skills of PhD and postdoctoral students"), project code POCU/380/6/13/123847/25.05.2019, SMIS code: 123847."

Under this grant, I achieved the following results:

1) I participated in the International Conference "Global Economy Under Crisis", organized by the Faculty of Economics at Ovidius University in Constanta, on December 10-11, 2020, where my paper entitled "Mathematics and the Prestige of the School Unit" was published
http://stec.univ-ovidius.ro/conferinte-detalii/the-international-conference-global-economy-under-crisis-9th-edition-december-10th-11th-2020

2) I participated in the 37th IBIMA Conference on April 1-2, 2021 in Cordoba, Spain where my paper titled "Managers Involved In The Development Of Educational Institutions" was published in the conference proceedings (ISBN:

978-0-9998551-6-4, Published in the USA).
https://ibima.org/accepted-paper/managers-involved-in-the-development-of-educational-institutions/

3) I have prepared a Business Plan for the establishment of an Educational Center called "After School and Before School FIBONACCI".

4) I participated in the European Integration - Realities and Perspectives (EIRP 2021) Conference, Galati, Romania, on May 14-15, 2021, with the oral presentation "Equations in algebras obtained by the Cayley-Dickson process.
https://zoom.us/j/91918263281?pwd=dk1MWGxJYUxVYlpwQ1lpYml Hc3gvUT09

5) I participated in the European Integration - Realities and Perspectives (EIRP 2021) Conference, held in Galati, Romania on May 14-15, 2021, with the oral presentation "Some solutions of the equation $x^2 + ax + b = 0$ with elements of type quaternion Fibonacci."
https://zoom.us/j/91918263281?pwd=dk1MWGxJYUxVYlpwQ1lpYml Hc3gvUT09

6) I participated in the Scientific Conference of Doctoral Schools SCDS-UDJG, Perspectives and Challenges in Doctoral Research, 9th Edition, Dunarea de Jos University of Galati, on 10-11 June 2021 with an oral presentation and poster entitled "Equations with Quaternions".
http://www.cssd-udjg.ugal.ro/index.php/abstracts-2022
http://www.cssd-udjg.ugal.ro/index.php/programme-21

As a PhD student at the university, I benefited from an "Excellence Scholarship for Research" during the period of July-September 2021, from the Doctoral School of Ovidius University in Constanța.

One of the most rewarding aspects of my academic journey has been the opportunity to contribute to existing knowledge through research. I am incredibly proud to be the author of several published articles that reflect my dedication and hard work. These articles not only represent the result of numerous hours of research and analysis but also demonstrate my growth as a researcher. They serve as milestones in my academic career:

1) I co-authored the Book chapter: Cristina Flaut, Diana Savin, **Geanina Zaharia**, *Some applications of Fibonacci and Lucas numbers*, in C. Flaut, S. Hoskova-Mayerova, F. Maturo, Book chapter in the Springer Book: Algorithms as a foundation for modern applied Mathematics, Springer, 2020, 119–130
https://link.springer.com/chapter/10.1007/978-3-030-61334-1$_5$
$https : //arxiv.org/abs/$1911.06863

In this chapter of the book, we present new applications of Fibonacci and Lucas numbers. We discover algebraic structures on certain sets defined with these numbers. We generalize the concept of Fibonacci and Lucas numbers by using an arbitrary binary relation over the real fields, rather than relying on the addition of real numbers, and develop properties for the newly obtained sequences.

Furthermore, by utilizing relations between Fibonacci and Lucas numbers, we propose a method to find new examples of split quaternion algebras and present new properties of these elements.

2) After attending the 8th International Eurasian Conference on Mathematical Sciences And Applications in Baku, Azerbaijan,where I presented the paper titled "Some Remarks Regarding Difference Equations of Degree n" from August 27-30, 2019. The conference website can be found at
http://www.iecmsa.org/organize/2019 and http://www.iecmsa.org/2019/. The improved work has been published in Mathematical Methods in the Applied Sciences: Cristina Flaut, Diana Savin, Geanina Zaharia, with the title "Properties and applications of some special integer number sequences" 44(2021), 7442-7454; DOI: 10.1002/mma.6257,
The full article can be accessed at
https://onlinelibrary.wiley.com/doi/full/10.1002/mma.6257.
The article was awarded in the Red zone in September 2021: PN-III-P1-1.1-PRECISI-2021- 56202, Position 1140 on the list: `https://bit.ly/43Mn61R`
In this article, we present the characteristics and applications of certain special integer sequences. We extend and provide several properties of the Pisano period. Additionally, we offer a new application in the field of cryptography and discuss the uses of quaternion elements.

3)Within the ANTREPRENORDOC project, I have developed the article titled "Remarks Regarding Computational Aspects in Algebras Obtained by Cayley–Dickson Process and Some of Their Applications", Mathematics 2022, 10, 1141. https://doi.org/10.3390/math10071141. However, since the publication occurred after the grant's conclusion, the publication fee was covered by the Doctoral School. Several new identities and properties are presented in the algebras obtained through the Cayley-Dickson process. As another remark regarding the computational aspects in these algebras, in the last part of this article, we solve quadratic equations in the real division quaternion algebra when the coefficients are special elements. These special coefficients allowed us to solve interesting quadratic equations, providing direct solutions without the need for specialized software.

4) In 2023, I have submitted the article titled "A Method for Solving quadratic Equations in Real Quaternion Algebra by using Scilab software" to the journal Italian Journal of Pure and Applied Mathematics (IJPAM-2023-76 is the manuscript reference number) for publication. The article has also been posted on arXiv: https://arxiv.org/abs/2304.08930
The article presents numerical applications of the quadratic equation with quaternionic coefficients, focusing on the algebra of real quaternions with parameterized coefficients denoted as $\mathbb{H}(\alpha, \beta)$. The specific values of α and β determine the properties and structure of the quaternion algebra. A new numerical algorithm is introduced for solving the equation $x^2 + ax + b = 0$ in the algebra $\mathbb{H}(\alpha, \beta)$.

The algorithm is developed based on well-known solving methods and utilizes the Scilab software for numerical computation. The advantages of using Scilab for solving quaternionic equations are highlighted. The article introduces a new numerical algorithm for solving quadratic equations with quaternionic coefficients in the algebra $\mathbb{H}(\alpha, \beta)$.

2.The current state of research in the field of Fibonacci numbers and cryptography.

In recent times, in several works, new properties and applications of certain sequences of integers have been presented, particularly in Cryptography and Coding Theory, such as the Fibonacci sequences, Lucas sequences, Pell sequences, $p-$Fibonacci sequences, Tribonacci sequences, and so forth.(see [Fl; 19], [HKN; 12], [Ko, Oz, Si; 17], [Me; 99], [Re; 13], [St; 06], [St; 07], etc.) I studied the specialized literature to see in which directions the researchers went and to which new horizons I can go. Using these works, we present our own results: several applications of special integer sequences in subsection 3.1.We extend the concept of Fibonacci numbers by utilizing arbitrary multiplication within the real number fields, as opposed to the addition of real numbers, and furnish properties of the sequences derived through this approach. Our own results are presented in subsection 4.1. Additionally, we generalize and provide some properties of the Pisano period; our own results are presented in subsection 3.3. Furthermore, we offer two applications in cryptography; our own results are presented in subsection 3.4.

Sir W. R. Hamilton was an Irish mathematician and physicist who made significant contributions to the field of mathematics, particularly in the areas of algebra and geometry. In the mid-19th century, he investigated the concept of quaternions, which are a type of extended complex numbers that have four real components greater than two. Hamilton's work on quaternions led him to discover complex quaternions as well. Approximately a century after Hamilton's discovery, A. F. Horadam introduced the concept of complex Fibonacci numbers and Fibonacci quaternions. These developments have inspired many researchers to make further contributions to the field of mathematics.

In a publication from 2008 ([Cr, Hr; 08]), the authors emphasize the crucial role played by the Golden ratio in modern physics, particularly in atomic physics, and in mathematical probability theory, with its connection to Fibonacci numbers. The article explores the significance of the Golden proportion in geometry and its applications in various fields, including nature, music, art, and physics. It also presents a complex version of the Golden number and studies its properties. Establishes connections between Golden structures and Riemannian metrics, a crucial geometric object in the study. I have researched from this material for my own result (Remark 4.1.7.) in subsection 4.1.

In a publication from 2013 ([Fl, Sh; 13]), the authors investigated the properties of generalized Fibonacci quaternions and Fibonacci-Narayana quaternions. These types of quaternions are extensions of the standard Fibonacci and Narayana sequences, and have been studied for their potential applications in various fields such as mathematics, computer science, and engineering. The authors presented a thorough analysis of the characteristics of these generalized quaternions, including their definitions, relations with other mathematical concepts, and ways in which they can be manipulated and utilized. This research provides valuable insights into the properties and potential uses of generalized Fibonacci quaternions and Fibonacci-Narayana quaternions.
In a series of publications, the authors have extensively studied generalized Fibonacci quaternions, Fibonacci-Narayana quaternions, and complex Fibonacci quaternions. Many of these results are presented in subsections 2.5. and 2.6.

In a publication from 2016 ([ET, SY, MS; 16]), the authors introduced the concept of bi-periodic Lucas quaternions, which generalized the standard Lucas quaternions. They provided the generating function and Binet formula for these quaternions, and establish relation between bi-periodic Fibonacci quaternions and bi-periodic Lucas quaternions. Many of these results are presented in subsections 2.7., 4.1. and 4.2.

In a 2019 publication ([FSZ; 19]), the authors discuss various applications of Fibonacci and Lucas numbers in algebraic structures and quaternion algebras. They explore algebraic structures defined using Lucas and Fibonacci numbers by employing an arbitrary binary relation over the real field instead of the traditional addition operation. This study brings to light properties and relations of the new sequences generated as a result of this generalization. Fibonacci and Lucas numbers are used to find new examples of quaternion algebras and to present novel properties of these mathematical elements. New applications of these sequences are offered through the study of algebraic structures and relations. New preliminary information about Fibonacci and Lucas numbers is presented, including their definitions and basic properties. Well-known formulas such as Binet's formula for the Fibonacci and Lucas sequences are mentioned. Properties of generalized Fibonacci-Lucas numbers, quaternion algebras, and various algebraic equations involving Fibonacci and Lucas numbers are described. The authors provide proofs and explanations for each result. The possibility of applying similar approaches to Fibonacci-type sequences is mentioned, and a second-degree difference equation is introduced. The authors also highlight the connection between the Fibonacci sequence and the golden ratio. New applications of Fibonacci and Lucas numbers in algebraic structures and quaternion algebras are explored, providing understanding of the properties and relations of these numbers and presenting new results in the field, in subsections 2.7. and 4.1.

In a publication from 2020 ([FLSZ; 20]) the authors showcased a cryptography application that utilizes recurrence relations akin to the Fibonacci sequence. Additionally, they introduced a new perspective on the Pisano period and its properties. In the exploration of algebras derived from the Cayley-Dickson process, determining whether an algebra is division or split holds significant significance. Moreover, if an algebra is split, it is interesting to see if we can find substructures that possess certain important properties for various applications such as cryptography, code theory, etc. Our own results from this publication can be identified in subsections 2.1., 3.1., 3.2., 3.3., 4.2..

In a publication from 2022 ([FZ; 22]) exploring algebras derived from the Cayley-Dickson process presents challenges in achieving desirable properties due to computational complexities. Hence, the discovery of identities within these algebras assumes significance, aiding in the acquisition of new properties and facilitating calculations. To this end, the study introduces several fresh identities and properties within the algebras derived from the Cayley-Dickson process. Additionally, when specific elements serve as coefficients, quadratic equations in the real division quaternion algebra can be solved, showcasing the authors ability to provide direct solutions without relying on specialized software. The own results from this article can be found in subsections 2.9.,3.4. and 5.1..

In 2023, in ([ZM; 23]) the authors address an innovative method for solving quadratic equations in real quaternion algebra. The method presented in the article focuses on identifying solutions to quadratic equations in the real division quaternion algebra. The authors develop a specific algorithm to obtain solutions in a precise and efficient manner by using the Scilab software, which provides a practical and applied approach for solving these equations with quaternions. Concrete examples are provided to illustrate the practical application of the proposed algorithm.

The significance of employing Scilab software in solving quadratic equations within the realm of real quaternion algebra is underscored in the article. The advantages of this tool and its valuable contribution to applied mathematics research are emphasized. Furthermore, the authors allude to potential extensions and advancements of the proposed method, paving the path for novel investigations and applications within the realm of quaternion algebra. The article makes a substantial contribution to the field of applied mathematics, offering researchers and students interested in quaternion algebra an efficient and practical method for solving quadratic equations. The own results from this article can be found in subsections 5.2., 5.3. and 5.4..

3.Work structure

In the first stage, we would like to highlight the fact that the new and original results of this thesis are obtained from the following articles or book chapters: [FSZ; 19], [FLSZ; 20], [FZ; 22], [ZM; 23].

My own results and those of my collaborations can be found in this thesis in Subsection 3.1 (Proposition 3.1.1, Proposition 3.1.2, Remark 3.1.3), Subsection 3.2 (Proposition 3.2.1, from Definition 3.2.3 to Example 3.2.7, inclusive), Subsection 3.3.1 (entire), my own Example 3.3.6, Subsection 3.4 (from Proposition 3.4.2 to Remark 3.4.7, inclusive), the entire Subsection 4.1 (from Example 4.1.1 to Remark 4.1.9, inclusive), Subsection 4.2 (from Proposition 4.2.4 to Proposition 4.2.12, inclusive), Subsection 5.1 (from Remark 5.1.2 to Proposition 5.1.10, inclusive), the entire Subsection 5.2 (from Proposition 5.2.1. to Corollary 5.2.4, inclusive), Subsection 5.3 (Algorithm source code), Subsection 5.4 (all examples from 5.4.1 to 5.4.11, inclusive).

My doctoral thesis is structured into six chapters. In the first chapter, I have presented the motivation that led me to pursue this doctorate in mathematics and shared the valuable experiences I have gained as a doctoral student. I have provided a list of conferences and scientific events in which I actively participated and presented papers related to my research. I provided a history of the study of the subject, articles that I have studied to familiarize myself with the properties, relations, and applications of these linear difference equations.

In the second chapter, titled *Preliminaries*, we explore the essential concepts and properties needed for our subsequent results. We define the characteristic equation and establish the recurrence relations for various sequences, including Fibonacci, Pell, Lucas, and Fibonacci-Narayana numbers. Additionally, we present important identities such as Cassini's identity, Catalan's identity, and Vajda's identity. This chapter contributes to understanding the algebras generated by the Cayley-Dickson process and highlights their computational aspects, providing insights into their properties and applications.

Chapter 3, titled *Special Integer Number Sequences and Some of Their Applications*, provides an in-depth exploration of special sequences of integers, their applications in coding theory and cryptography, the concept of Pisano's period, encryption and decryption processes using difference equations, and the study of algebras obtained through the Cayley-Dickson process.

In section 3.1., it has three distinct original results:

1) In Proposition 3.1.1, I present three relations between the matrix associated with the recurrence sequence D_k and the matrix Y_i. Y_i is formed by a single row and k columns, generated by the terms d_i of the same recurrence sequence.

2) In Proposition 3.1.2, using the same notations, for the recurrence relation of degree n, we have identified three true relations.

3) In Remark 3.1.3, I show that relation ii) is a extension of the Cassini identity, and in relation iii), we obtain a relations for Fibonacci numbers by choosing $k = 2$ and $a_1 = 1$, respectively $k = 3$.

In Section 3.2, we have the following own results:

1) In Proposition 3.2.1, I have reported that the sequence generated by d_n when considered modulo m is periodic, and I have also provided the proof of this proposition.

2)In Definition 3.2.3, I have introduced $\pi(m)$, denoted as the Pisano period, for the sequence (d_m). It represents the frequency with which the sequence (d_n) repeats itself modulo m.

3)Remark 3.2.4. refers to the properties of the matrix D_k, which is associated with the difference equation in (3.1.1), and how it is related to the Pisano period of the sequence (d_n) modulo m.

The matrix D_k is a specific matrix associated with the difference equation (3.1.1). $\pi(m)$ represents the Pisano period of the sequence (d_n) when taken modulo m. I_k is the identity matrix of order k. The first equation, $D_k^{\pi(m)} \equiv I_k \ mod \ m$, shows that the matrix D_k raised to the power of $\pi(m)$ is congruent to the identity matrix I_k modulo m. The second equation, $((-1)^{(k+1)} a_k)^{\pi(m)} \equiv 1 \ mod \ m$, indicates that the expression $((-1)^{(k+1)} a_k)^{\pi(m)}$ is congruent to 1 modulo m. The last two equations, $ord\left((-1)^{(k+1)} a_k\right) \mid \pi(m)$, show that the order of $(-1)^{(k+1)} a_k$ (i.e., the smallest positive power for which $(-1)^{(k+1)} a_k$ becomes congruent to 1 modulo m) must be a divisor of $\pi(m)$. This means that the Pisano period $\pi(m)$ is closely related to the properties of $(-1)^{(k+1)} a_k$ modulo m.

4) Theorem 3.2.5 discusses several properties of the Pisano period $(\pi(m))$ in the context of sequences generated by recurrence equations. This theorem demonstrates how the Pisano period behaves in various situations, providing valuable insights into the repetition of values in sequences generated by recurrence equations:

i) If an integer s_1 is divisible by another integer s_2, then the Pisano period for s_1 $(\pi(s_1))$ must be a divisor of the Pisano period for s_2 $(\pi(s_2))$. In other words, if a sequence has a shorter period when working with a certain modulus, then the same sequence will have a longer period when working with a multiple of that modulus.

ii) The Pisano period for the least common multiple of two numbers, s_1 and s_2, is equal to the least common multiple of their individual periods $(\pi(s_1)$ and $\pi(s_2))$.

iii) For prime numbers raised to a power (p^r), the Pisano period for p^{r+1} is either equal to the Pisano period for p^r or is p times the Pisano period for p^r. This depends on the properties of the matrix associated with the recurrence equation.

iv) If the Pisano period for p^r is different from the Pisano period for p^{r+1}, then the Pisano period for p^{r+1} is also different from the Pisano period for p^{r+2}. This suggests that the Pisano periods for consecutive powers of the same prime number are related in a specific way.

v) If the matrix associated with the recurrence equation is diagonalizable, and the prime number p does not divide a specific coefficient of the equation, then the Pisano period for p must be a divisor of $p-1$. This property is related to the properties of the associated matrix.

vi) If we have a sequence generated by a recurrence equation with specific initial conditions, and all the coefficients of the equation are odd numbers, then the Pisano period for modulus 2 ($\pi(2)$) is equal to $k+1$, where k is the order of the recurrence equation.

5)Example 3.2.7. addresses various situations related to calculating the Pisano period ($\pi(m)$) in the context of recurrence equations. This example illustrates how the Pisano period is calculated in different scenarios and how it depends on the coefficients of the recurrence equations as well as the prime number p.

i) Consider the recurrence equation with $k = 3$ and odd coefficients a_1, a_2, a_3. The Pisano period for this equation modulo 2 is calculated, showing that $\pi(2) = k + 1 = 4$.

ii) It is noted that for even coefficients a_i, the calculation of $\pi(2)$ becomes more complicated and depends on the number and position of the even coefficients. An example is provided with $k = 3$, where a_1 and a_3 are odd, and a_2 is even. In this situation, $\pi(2) = 7$.

iii) An example is taken with coefficients and $p = 3$. The value of $\pi(3)$ is computed and shown to be 6. Then, the calculation of $\pi(9)$ is performed, also yielding a value of 6. Using Theorem 3.2.5, iii), it is concluded that $\pi(27) = 18$.

iv) The calculations are carried out in $\mathbb{Z}_5$, where $p = 5$. Coefficients $a_1 = 6$, $a_2 = -11$, and $a_3 = 6$ are considered. The value of $\pi(5)$ is calculated and shown to be 4.

In Section 3.3.1 titled "CryptoMatrix", I present an original result:

1) The application of difference equations in cryptography involves encoding text messages using matrices derived from a specific sequence. The encryption key is generated based on properties related to the sequence, enabling secure message encryption. The decryption process utilizes the same key, making it efficient and secure, with the Pisano period playing a significant role in key variation for enhanced security.

2) Example 3.3.1. is an original result where I apply difference equations in cryptography. I introduce a encryption method based on matrices generated from a specific sequence, using the encryption key $(3, 2, 1, 1, 1, 3)$. Matrix operations are detailed for encryption, which converts the plaintext "ABBAAB" into the encrypted text "BBAABB." The example also highlights the role of the Pisano period in key variation for enhanced security and describes the decryption process

using the same key.

3) Example 3.3.2. is an original result demonstrating the application of difference equations in cryptography. It utilizes a specific matrix generated from a given sequence and an encryption key $(3, 27, 4, -5, 2, 2)$. The plain text "$SUCCESS**$" is transformed into the encrypted text "$HDSBSCKVS$". This example highlights the complexity of the method, where the same letters can be encrypted into distinct characters, and dissimilar characters can be encoded as the same letter, making encrypted texts challenging to decipher. The decryption process uses the same key as encryption, and the Pisano period (18) plays a significant role in key determination for data security.

4) Remark 3.3.3. emphasizes that the cryptosystem presented in this section is of symmetric type, using the same key for encryption and decryption. To address the secure key transmission issue, it suggests the use of hybrid cryptosystems, combining symmetric and asymmetric methods, or the encryption of symmetric keys with special encryption keys, distributed through a complex algorithm.

In Section 3.3.2. titled "CryptoKeyLock", I present an original result:

Example 3.3.6 introduces a concept called "complete sequence" and presents a method for encrypting and decrypting messages based on this complete sequence. To do this, Definition 3.3.4 is utilized, where "g-complete" is defined, and Theorem 3.3.5 is also employed, stating that the sequence of positive integers $(d_m)_{m\geq 0}$ generated by the given difference equation in the example is g-complete.

Example 3.3.6 illustrates this theory in the context of encryption and decryption. In this example, an alphabet with 27 letters (A, B, C, ..., Z, x) is used, where "x" represents a black space. The message "MATHxISxBEAUTIFUL" is encrypted. The message is divided into equal-length blocks: MATH/xISx/BEAU/TIFU/Lxxx. Coefficients for the difference equation are chosen to generate a complete sequence, resulting in an encryption key. Each block is then transformed into a string of encrypted letters using this key.

The generation of each encrypted block follows these steps:

- Coefficients for the difference equation are chosen, and terms d_k, d_{k+1}, ..., d_q are calculated for the number associated with the block using the presented theorem. - An encryption key is derived, containing the coefficients of the difference equation and the length of the encrypted block.

- For each block, coefficients c_{iq}, $c_{i(q-1)}$, ..., $c_{i(k-1)}$ are computed according to the equation presented.

- The encrypted string is obtained based on the values of the coefficients. If a coefficient is greater than 27 (the number of letters in the alphabet plus black space), it is converted into a special character in the encrypted text.

- To decrypt, the corresponding decryption key for each block is used to obtain the original blocks.

The example provides detailed calculations for the encrypted message blocks and the decryption process, demonstrating the functioning of the encryption and decryption method based on complete sequences.

In Section 3.4, we have the following original results:

1) In Proposition 3.4.2, several identities and relations are established for the algebras $\mathcal{A}_t$ obtained through the Cayley-Dickson process. These are valid for all algebras $\mathcal{A}_t$ obtained through this procedure, regardless of the characteristic of the field $\mathbb{K}$. The proof of these identities is based on the linearization method presented in reference [BH; 01]. This method expresses the given relations in the form of a linear expression, taking into account the specific transformations of the elements in these algebras.

2) Remark 3.4.3 explains that the mentioned results hold true for fields of arbitrary characteristics, including when char$\mathbb{K} \neq 2$. However, when char$\mathbb{K} \neq 2$, the proof of the relation (3.4.2) becomes simpler.

3)In Proposition 3.4.4, it was established that in all algebras $\mathcal{A}_t$ obtained through the Cayley-Dickson process, the following identity holds:

$$(x^m, y, x^n) = 0, m, n \in \mathbb{N};$$

The proof of this result relies on the algebraic properties of the algebras $\mathcal{A}_t$.

4) In Proposition 3.4.5, it was asserted that if we have an algebra $\mathcal{A}_t$ obtained through the Cayley-Dickson process and elements x and y satisfying the following equality:$(x\overline{y} + y\overline{x})^2 = \gamma^2 \mathbf{n}(x)\,\mathbf{n}(y), \gamma \in \mathbb{K} - \{0\}$.

Then, the algebra generated by x and y, denoted as $< x, y >$, has dimension 1. This statement was demonstrated using the properties of elements in the algebra $\mathcal{A}_t$.

5)In Proposition 3.4.6, the following was demonstrated:

(1) If we have an element a in the algebra $\mathcal{A}_t$, and $a \in \mathcal{A}_t - \mathbb{K}$, then the solutions in $\mathcal{A}_t$ of the equation $x^2 = a$ are the solutions in K of the following system:

$$\begin{cases} 2x_0^2 - \mathbf{n}(x) = a_0 \\ 2x_0 x_i = a_i, i \in \{1, 2, \ldots, 2^t - 1\} \end{cases},$$

where $a = \sum_{i=0}^{2^t-1} a_i f_i, x = \sum_{i=0}^{2^t-1} x_i f_i, a_i, x_i \in K.$

(2) If $a \in \mathbb{K}$, then the solutions in $\mathcal{A}_t$ of the equation $x^2 = a$ are the solutions in $\mathbb{K}$ of the following system:

$$\begin{cases} 2x_0^2 - \mathbf{n}(x) = a \\ 2x_0 x_i = 0, i \in \{1, 2, \ldots, 2^t - 1\} \end{cases}.$$

6) Remark 3.4.7 explains that if we have an element a in the algebra $\mathcal{A}_t$, it can be written as $a = a0 + \overrightarrow{a}, a_0 \in \mathbb{K}$. Therefore, all elements b in $\mathcal{A}_t$ of the form $b = \tau + \theta\overrightarrow{a}, \tau, \theta \in \mathbb{K}$, commute with a.

In Chapter 4 titled *Some applications of special elements defined over quaternion algebras*, we will study the properties of Fibonacci and Lucas sequences, focusing on Fibonacci numbers and their relations. We will explore the properties and applications of these special sequences of integers, with a particular emphasis on quaternion algebra. We will analyze the recurrence relations, Binet formulas, modulo properties, and the rings and modules generated by these numbers. Various properties of these special sequences of integers will be presented.

In section 4.1, I present the following original results:

1) Example 4.1.1: This presents specific examples of Fibonacci numbers, including f_{10}, f_{15}, and f_{20}, and observes that they are related to factors of other smaller Fibonacci numbers.

2) Proposition 4.1.2: It states that the set A, defined as all numbers of the form αf_{5n}, where α and n are integers, forms an ideal in the ring of integers.

3) Proposition 4.1.3: This refers to generalized Fibonacci-Lucas numbers and asserts that a certain set containing these numbers is also an ideal in the ring of integers.

4) Proposition 4.1.4: It asserts properties of certain quaternion algebras (algebras based on the quaternion imaginary unit) related to Fibonacci and Lucas numbers, such as these algebras being decomposed over rational numbers.

5) Proposition 4.1.5: This discusses Fibonacci sequences defined by different addition relations, using a binary relation $*$ instead of addition. It provides a formula for calculating the elements of these sequences and mentions a limit based on the roots of the equation associated with the $*$ relation.

6) Proposition 4.1.6: This proposition presents properties of specific quaternion algebras (algebras based on the imaginary unit quaternion) related to Fibonacci and Lucas numbers, such as the fact that these algebras are split algebras over rational numbers.

7) Remark 4.1.7: This adds clarifications and additional observations to Proposition 4.1.6.

8) Proposition 4.1.8: This proposition addresses properties of generalized Fibonacci-Lucas numbers and asserts that a certain set containing these numbers is also an ideal in the ring of integers.

9) Remark 4.1.9: This adds clarifications and additional observations to Proposition 4.1.8.

In section 4.2., I present the following original results:

1) Proposition 4.2.4: The following relationships have been established: i) If l is a prime number, then l divides a_n only when n is an even number. ii) $a_n \equiv 1 \pmod{l^2}$ if and only if n is an odd number.

2)Proposition 4.2.5: It has been proven that the set $M = \alpha a_{2n} | n \in \mathbb{N}, \alpha \in \mathbb{Z}$ forms a non-unitary commutative ring with addition and multiplication operations.

3)Proposition 4.2.6: It has been demonstrated that the set M serves as a two-sided ideal within the ring $(\mathbb{Z},+,\cdot)$, specifically, $M=l\mathbb{Z}$.

4)Proposition 4.2.7: It has been established that all l-quaternion numbers are invertible within the quaternion algebra $\mathbb{HZ}l\,(-1,-1)$.

5)Proposition 4.2.8: It has been shown that all l-quaternion numbers are invertible within the quaternion algebra $\mathbb{HZ}l^r\,(-1,-1)$..

6) Proposition 4.2.9: A new relations has been presented: $a_n+a_{n+3\cdot 2^k}=M_k\left(M_k^2-3\right)a_{n+3\cdot 2^{k-1}}$, for $k\geq 2$.

7) Proposition 4.2.10: It has been proven that l-quaternions satisfy the relations $A_n=A_{n+2}$.

8) Proposition 4.2.11.: The invertibility of "l-quaternions" in quaternion algebras was proven.

9) Proposition 4.2.12.: Properties of "l-quaternions" in quaternion rings and modules were explored, including specific relations between them.

In Chapter 5 titled *Equations over quaternions and octonions*, we will present specific formulas for solving the quadratic equation $x^2+bx+c=0$, with $b,c\in\mathbb{H}(-1,-1)$, in different cases. We will provide specific examples of quadratic equations and their solutions in the context of Fibonacci quaternion algebras. We will discuss division algebras and solutions of second-degree equations in the algebra $\mathbb{H}(\alpha,\beta)$. It is explained that the algebra $\mathbb{H}(\alpha,\beta)$ is a mathematical construction whose properties depend on the chosen values for α and β. We will provide step-by-step calculations and formulas for obtaining the solutions of quadratic equations.

In Section 5.1, I present the following original results:

1) Remark 5.1.2:

- I investigated whether Δ is zero or not.
- I considered a particular case with m=n=3.
- If we use two quaternions, we fall into subcases 3 and 4 of Proposition 5.1.1. To obtain the solutions, we need to find the roots of the cubic polynomial.

2) Proposition 5.1.3: I presented the solutions of equation (5.1.11).

3) Proposition 5.1.4: I presented the solutions of equation (5.1.12).

4) Proposition 5.1.5: I presented the solutions of equation (5.1.13).

5) Proposition 5.1.6: I presented the solutions of equation (5.1.19).

In Section 5.2, I present the following own results:

1) Proposition 5.2.1: I present the solution to the monic equation $x^2+bx+c=0$ in the context of the algebra $\mathbb{H}(\alpha,\beta)$ where b and c are two quaternions from this algebra. This also enables the determination of the quaternion values x_1,x_2,x_3,x_4.

2) Corollaries 5.2.2, 5.2.3, 5.2.4: These are statements derived from Theorem 5.1.1.

In sections 5.3 and 5.4, we have our own result, and we include an algorithm for implementing calculations and numerical applications and examples to demonstrate the practical use of the formulas and algorithm. We will provide several specific examples, illustrating the calculations and solutions for various quadratic equations in the algebra $\mathbb{H}(\alpha, \beta)$.

Chapter 6 consists of *Conclusions*, followed by *Further researches*, *List of own publications* and *65 References*.

The new and original results presented by me in this doctoral thesis have been published in the following articles or book chapter: [FSZ; 19], [FLSZ; 20], [FZ; 22], [ZM; 23].

4.Acknowledgements

This work is partialy supported by the project ANTREPRENORDOC, in the framework of Human Resources Development Operational Programme 2014-2020, financed from the European Social Fund under the contract number 36355/23.05.2019 HRD OP /380/6/13 – SMIS Code: 123847.

5.List of own publications

1) Cristina Flaut, Diana Savin, **Geanina Zaharia**, „Properties and applications of some special integer number sequences", Mathematical Methods in the Applied Sciences, 44(2021), 7442-7454;
DOI:10.1002/mma.6257,
https://onlinelibrary.wiley.com/doi/full/10.1002/mma.6257
https://arxiv.org/pdf/2001.03963.pdf, Red Zone

2) Cristina Flaut and **Geanina Zaharia** „Remarks Regarding Computational Aspects in Algebras Obtained by Cayley–Dickson Process and Some of Their Applications", Mathematics 2022, 10 (7), 1141.
https://doi.org/10.3390/math10071141
https://www.mdpi.com/2227-7390/10/7/1141

3)Cristina Flaut, Diana Savin and **Geanina Zaharia** „Some Applications of Fibonacci and Lucas Numbers", capitol Springer: Algorithms as a Basis of Modern Applied Mathematics, ISSN 1434-9922 ISSN 1860-0808 (electronic) Studies in Fuzziness and Soft Computing ISBN 978-3-030-61333-4 ISBN 978-3-030-61334-1 (eBook) https://doi.org/10.1007/978-3-030-61334-1 https://arxiv.org/abs/1911.06863

4) **Geanina Zaharia** and Diana Munteanu "A Method for Solving quadratic Equations in Real Quaternion Algebra by using Scilab software" to the journal Italian Journal of Pure and Applied Mathematics (IJPAM-2023-76 is

the manuscript reference number) for publication. The article has also been posted on arXiv: https://arxiv.org/abs/2304.08930

6.REFERENCES:

1.[A; 80] Atkinson, A.C., *Tests of pseudo-random numbers*, Applied Statistics, 29(1980), 164-171.

2.[AW; 97] Agarwal, R.P., Wong, P.J.Y., *Advanced Topics in Difference Equations*, Springer Netherlands, 1997, 510 p.

3.[B; 61] Brown, Jr.J.L., *Note on Complete Sequences of Integers*, The American Mathematical Monthly, 68(6)(1961), 557-560.

4.[BH; 01] Bremner, M., Hentzel, I.*Identities for Algebras Obtained from the Cayley-Dickson Process. Commun. Algebra*, 2001, *29*, 3523-3534.

5.[CH; 98] Cho, E., *De-Moivre's formula for quaternions*, Appl. Math. Fiet., 11 (6) (1998), 33-35.

6.[Ch; 20] Chapman, A., *Factoring Octonion Polynomials. Int. Algebra Comput.*, 2020, *30*, 1457-1463.

7.[Ch; 20(1)] Chapman, A. *Polynomial Equations over Octonion Algebras. J. Algebra Appl.* 2020, *19*, 2050102.

8.[Cr, Hr; 08] Crasmareanu, M., Hretcanu, C., *Golden differential geometry*, Chaos Solitons and Fractals, 38(5)(2008), 1229-1238.

9.[Di, St; 03] Didkivska, T.V., St'opochkina, M.V., *Properties of Fibonacci-Narayana numbers*,
In the World of Mathematics, 9(1)(2003), 29–36. [in Ukrainian]

10.[EPC; 18] European Payments Council, *Guidelines on Cryptographic Algorithms Usage and Key Management*, 2018

11.[ET, SY, MS; 16] Elif, T., Semih, Y., Murat, S., *A note on bi-periodic Fibonacci and Lucas quaternions*, Chaos Solitons & Fractals V.85,(2016) pages 138-142

12.[Fib.] http://www.maths.surrey.ac.uk/hosted-sites/R.Knott/Fibonacci/fib.html

13.[Fl, Sh; 13] Flaut, C., Shpakivskyi,V., *On Generalized Fibonacci Quaternions and Fibonacci-Narayana Quaternions*, Adv. Appl. Clifford Algebras, 23(3)(2013), 673-688.

14.[FS; 13] Flaut, C., Shpakivskyi, V., *Some identities in algebras obtained by the Cayley-Dickson process. Adv. Appl. Clifford Algebr.*, 2013, *23*, 63-76.

15.[FSH; 15] Flaut, C., Shpakivskyi, V., *An Efficient Method for Solving Equations in Generalized Quaternion si Octonion Algebras*, Adv. Appl. Clifford Algebras, 25(2)(2015), 337-350.

16.[FS; 18] Flaut, C., Savin, D., *Some special number sequences obtained from a difference equation of degree three*, Chaos, Solitons & Fractals, 106(2018), 67-71.

17.[Fl; 19] Flaut, C., *Some application of difference equations in Cryptography and Coding Theory*, Journal of Difference Equations and Applications, 25(7)(2019), 905-920. https://doi.org/10.1080/10236198.2019.1619713

18.[Fl, Sa; 19] Flaut, C., Savin, D., *Some remarks regarding l-elements defined in algebras obtained by the Cayley-Dickson process*, Chaos, Solitons & Fractals, 118(2019), 112-116.

19.[FSZ; 19] Flaut, C., Savin, D., **Zaharia, G.**, *Some applications of Fibonacci and Lucas numbers*, in Flaut,C., Hoskova-Mayerova,S., Maturo,F., Book chapter in the Springer Book: Algorithms as a foundation for modern applied Mathematics, Springer, 2020, 119–130
https://link.springer.com/chapter/10.1007/978-3-030-61334-1 5
https://arxiv.org/abs/1911.06863

20.[FLSZ; 20] Flaut, C., Savin, D., **Zaharia, G.**, *Properties and applications of some special integer number sequences*, Mathematical Methods in the Applied Sciences, 44(2021), 7442-7454;
https://onlinelibrary.wiley.com/doi/full/10.1002/mma.6257

21.[FZ; 22] Flaut, C., **Zaharia, G.** *Remarks Regarding Computational Aspects in Algebras Obtained by Cayley–Dickson Process and Some of Their Applications*, Mathematics 2022, 10, 1141.
https://doi.org/10.3390/math10071141
https://www.mdpi.com/2227-7390/10/7/1141

22.[Gi, Sz; 06] Gille, P., Szamuely, T., *Central Simple Algebras and Galois Cohomology*, Cambridge University Press, 2006.

23.[HKN; 12] Han, J.S., Kim, H.S., Neggers, J., *Fibonacci sequences in groupoids*, Advances in Difference Equations 2012, 2012:19

24.[Ho; 61] Horadam, A. F., *A Generalized Fibonacci Sequence*, Amer. Math. Monthly, 68(1961), 455-459.

25.[Ho; 63] Horadam, A. F., *Complex Fibonacci Numbers si Fibonacci Quaternions*, Amer. Math. Monthly, 70(1963), 289-291.

26.[HS; 02] Huang,L., So,W., *Quadratic Formulas for Quaternions*, Applied Mathematics Letters, 15(2002), 533-540.

27.[He; 97] Hentzel, I.R. *Identities of Cayley-Dickson Algebras. J. Algebr.*, 1997, *188*, 292-309.

28.[Is; 84] Isaev, I.M., *Identities of a finite Cayley-Dickson Algebra. Algebra i Logika*, 1984, *23*, 407-418.

29.[Jo] Johnson, R.C., *Fibonacci numbers and matrices*, Available at http://maths.dur.ac.uk/dma0rcj/PED/fib.pdf.

30.[JO; 10] Janovska, D., Opfer, G.*A note on the computation of all zeros of simple quaternionic polynomials. Siam J. Numer. Anal.*, 2010, *48*, 244-256.

31.[Ku; 99] Kuipers, J.B., *Quaternions and Rotation Sequences: A Primer with Applications to Orbits, Aerospace, and Virtual Reality*, Princeton University Press, 1999.

32.[Ko; 94] Koblitz, N., *A Course in Number Theory and Cryptography*, Springer Verlag, New-York, 1994, p.65-76.

33.[Ko, Oz, Si; 17] Koroglu, M.E., Ozbek,I., Siap,I., *Optimal Codes from Fibonacci Polynomials si Secret Sharing Schemes*, Arab. J. Math, 2017, 1-12.

34.[L; 01] Lenstra, A.K, Verheul, E.R., *Selecting Cryptographic Key Sizes*, J. Cryptology, 14(2001), 255-293.

35.[Mag.] Magma Computational Algebra System http://magma.maths.usyd.edu.au/magma/handbook/

36.[MAk, HK, MT; 14] Mahmut, A., Hidayet, H. K., Murat, T., *Fibonacci Generalized Quaternions*, Advances in Applied Clifford Algebras volume 24,(2014),pages 631–641

37.[Me; 99] Melham, R., *Sums Involving Fibonacci and Pell Numbers,* Portugalie Mathematica, 56(3)(1999), 309-317.

38.[MS; 08] Mierzejewski, D. A., Szpakowski, V. S., *On solutions of some types of quaternionic quadratic equations*, Bull. Soc. Sci. Fiet. Lódź, 58, Ser. Rech. Déform.,55 (2008), 49-58.

39.[Lu; 04] Ludkovsky, S.V., *Zeros of Polynomials Over Cayley-Dickson Algebras*. Available online: `https://arxiv.org/pdf/math/0406048.pdf` (accessed on 06.07.2023)

40.[Ni; 41] Niven, I.,*Equations in quaternions. Am. Math. Monthly* **1941**, *48*, 654-661.

41.[Po; 97] Porter, R.M.,*Quaternionic Linear and Quadratic Equations. J. Nat. Geom.* ,1997, *11*, 101-106.

42.[PRD; 10] Pogoruy, A., Rodrigues-Dagnino, R.M., *Some algebraic and analytical properties of coquaternion algebra*, Adv. Appl. Clifford Alg., 20(2010), 79-84.

43.[PW; 02] Pumplün, S., Walcher, S.*On the zeros of polynomials over quaternions. Commun. Algebra*, 2002, *30*, 4007-4018.

44.[Ra; 88] Racine, M.L., *Minimal Identities of Octonions Algebras. J. Algebr.*, 1988, *115*, 251-260.

45.[Re; 13] Renault, M., *The Period, Rank, and Order of the (a,b)-Fibonacci Sequence Mod m*, Mathematics Magazine, 86(5)(2013), 372-380, https://doi.org/10.4169/math.mag.86.5.372.

46.[Sa; 17] Savin, D., *About special elements in quaternion algebras over finite fields*, Advances in Applied Clifford Algebras, vol. 27, June 2017, Issue 2 , p. 1801- 1813.

47.[Sa; 19] Savin, D. *Special numbers, special quaternions and special symbol ele- ments*, chapter in the book Models and Theories in Social Systems, vol. 179, Springer 2019, ISBN-978-3-030-00083-7 , p. 417-430.

48.[St; 06] Stakhov, A.P., *Fibonacci matrices, a generalization of the "Cassini formula", and a new coding theory*, Chaos, Solitons and Fractals, 30(2006), 56-66.

49.[St; 07] Stakhov, A.P., *The "golden" matrices and a new kind of cryptography*, Chaos, Solitons and Fractals, 32(2007), 1138-1146.

50.[Sc; 66] Schafer, R.D., *An Introduction to Nonassociative Algebras*; Academic Press: New York, NY, USA, 1966.

51.[Sc; 54] Schafer, R.D., *On the algebras formed by the Cayley-Dickson process. Amer. J. Math.*, 1954, *76*, 435-446.

52.[SM; 04] Smith, W.D., *Quaternions, octonions, si now, 16-ons, and 2n-ons; New kinds of numbers*, www. math. temple.edu/ 2dcwds/homepage/nce2.ps, 2004.

53.[SS; 01] Serodio, R., Siu, L.S., *Zeros of Quaternion Polynomials. Appl. Math. Lett.*, 2001, *14*, 237-239.

54.[SPV; 01] Serodio, R., Pereira, E., Vitoria, J., *Computing the Zeros of Quaternion Polynomials. Comput. Math. Appl.*, 2001, *42*, 1229-1237.

55.[TK; 01] Koshy, T., *Fibonacci and Lucas Numbers with Applications*, John Wiley Sons, 2001
https://books.google.ro/books?
id=1iDKKceqD2sCprintsec=frontcoverhl=rov=onepageqf=false (accessed on 30.06.2023).

56.[Wa; 60] Wall, D.D., *Fibonacci Series Modulo m*, The American Mathematical Monthly,67(6)(1960), 525-532.

57.[WWW1] http://www.maths.surrey.ac.uk/hosted-sites/
R.Knott/Fibonacci/fib.html

58.[WWW2] http://www.maths.surrey.ac.uk/hosted-sites/
R.Knott/Fibonacci/lucasNbs.html

59.[WWW3] http://mathworld.wolfram.com/PellNumber.html

60.[WWW4] https://study.com/academy/lesson/how-to-prove-cassinis-identity.html

61.[WWW5] https://mathworld.wolfram.com/CatalansIdentity.html

62.[WWW6] https://fmi.univ-ovidius.ro/wp-content/uploads/2021/scm/abstractsB.pdf

63.[Z; 72] Zeckendorf, E., *Representation des nombres naturels par une somme de nombres de Fibonacci ou de nombres de Lucas*, Bull. Soc. R. Sci. Lićge, 41(1972), 79-182.

64.[Zo; 41] Zorn, M., *Alternative rings and related questions I: Existence of the radical. Ann. of Math.* 1941, *42*, 676-686.

65.[ZM; 23] **Zaharia, G.**, Munteanu, D., *A Method for Solving quadratic Equations in Real Quaternion Algebra by using Scilab software*, to the journal Italian Journal of Pure and Applied Mathematics (IJPAM-2023-76 is the manuscript reference number) for publication, accepted: 10.08.2023
The article has also been posted on arXiv:
http://arxiv.org/abs/2304.08930