"OVIDIUS" UNIVERSITY FROM CONSTANTA

DOCTORAL SCHOOL OF MATHEMATICS

MATHEMATICS DOCTORAL DOMAIN

Phd Thesis

Special classes of algebras and some of their applications

PhD Coordinator
Prof. univ. dr. habil. Cristina Flaut

PhD Student
Radu Vasile

CONSTANTA, September 2021

Contents

Chapter 1

Introduction

1.1 Motivation

Algebraic structures play an important role in many practical applications. Cryptography ([33]), error-correcting codes theory ([30], [11]), artificial intelligence ([14]) are just a few examples of such applications. A special interest has recently been aroused by algebraic structures called "algebras". Under this umbrella term, many types of algebraic structures are included, being both algebraic structures defined as modules or vector spaces over a field, such as algebras obtained by the Cayley Dickson process ([41]), or logical algebras, which are sets with certain operations that respect certain axioms (some examples can be found in [24], [25], [20], [7], [1]). So, starting from the term "algebra", we can divide these structures into several classes, hence the title of this work: "Special classes of algebras".

In recent developments in wireless communication, the problem of efficient coding has become increasingly complicated, especially after the introduction of multiple antenna communication both at transmission and reception (MIMO Channels). The first such a code, based on division algebras, was constructed by Alamouti in 1998 and presented in [2]. In the article, Alamouti uses the algebra of real quaternions, taking its representation over a maximum subfield (in the sense of inclusion), namely over $\mathbb{C}$, the set of complex numbers, building a Codebook with all the necessary properties. In [37] it is shown how division algebras can be important tools in building codes for such situations. More precisely, it is shown how a specific class of division algebras, cyclic division algebras, can be used. In this article, division algebras are assumed to be non-commutative from the beginning, by definition. In [44] the class of cyclic division algebras is restricted to generalized quaternion and biquaternion algebras. Although these are particular cases of cyclic algebras, they have an important property. They have attached a quadratic form that can be used to determine when an algebra is a division algebra or when it is a split one which is quite difficult to determine otherwise. Although this article also talks about commutative division algebras, the authors say that they are only interested in those non-commutative algebras, the reason being simple, by construction, these non-commutative division algebras have the property of producing matrices with the fully diverse property, i.e. the difference between any two matrices has a not null determinant.

In [38], a step further is taken, and non-associative algebras are constructed and studied in terms of code theory. Also, in this paper, those elements of an algebra that have a richer set of properties are exploited in an interesting way, such as the nucleus of non-associative algebra, i.e. all those elements that associate with any other element in algebra. With their help, it is shown how to construct a representation over a maximum subfield of such an algebra, thus obtaining error-correcting codes.

On the other hand, in [26] a spectacular result is presented, obtaining a binary block code starting from a BCK-algebra. In this paper, the authors launch a challenge by asking whether it is possible to construct a BCK-algebra from a binary block code. Responding to the challenge, in [15] the author constructs a BCK-algebra starting from the binary code used in [26]]. It is found, that the two BCK-algebras associated with the same code are different and do not have common properties. Thus, a new property of two BCK-algebras is established, that of being "Code similar". The topic about connections between binary block codes and BCK-algebras is also developed in [39] and in [5] the authors introduce n-ary block codes attached to BCK-algebras. Starting from BCK algebras, various generalizations have been made, resulting in defining new various types of logical algebras: BE-algebras [28], Dual BCK-algebras [29] and others [23]. On the other hand, equivalence relations have been established between various other classes of logical algebras and BCK-algebras. Based on this research, but also on the equivalence relations between various classes of logical algebras, a question arises: "Can error-correcting codes be attached to other types of logical algebras"? A first answer is given in [31], where the author attaches binary error correction codes to Hilbert algebras, using the equivalence between Hilbert algebras and positive implicative BCK-algebras. Based on this extension, we wanted to investigate whether binary block codes can be attached to other types of logical algebras. At the same time, starting from the answers found for this first question, other interesting questions arose: "Are these codes usable in their current form"? "Who influences whom? Do codes influence the properties of logical algebras? Or, do the properties of logical algebras influence, in a positive sense, the construction of good codes?" "Can we find new interesting things about logical algebras by researching the attached codes?" And many others who will find answers in the pages of this work.

1.2 Main results

First of all, we mention that the new and original results of this work come from the following articles or book chapters: [17], [18], [19], [42], [43]. In addition to these results, for a brief introduction to the world of error-correcting codes, the reader can consult the article from [16].

1.2.1 Binary block codes attached to Wajsberg and MV-algebras

The purpose of this work is to present a research about special classes of algebra and about some new applications of them. In particular, observing the results in [26] (presented in 5.1), we asked ourselves whether those results can be transferred to other classes of logical algebras, given the equivalence relations between them. In [19], we built a mathematical apparatus similar to the one in [26] and thus attached binary error correcting codes to the Wajsberg and MV-algebras (see 5.2.1 and 5.2.2). In the case of MV algebras, we also constructed the distance function attached to such a code, showing that it is equivalent to the known Hamming distance (see 5.2.13). After attaching these codes, we showed that, given a commutative and bounded BCK algebra, together with the associated MV and Waisberg algebras, the same error-correcting code is obtained (Theorem 5.2.25). Moreover, we have shown that this binary error correction code acts as a template on which we can place any of the three types of algebras (see 5.2.27).

1.2.2 Number of Wajsberg algebras and their representation

Studying the codes attached to MV and Wajsberg algebras, we found a simple way to count Wajsberg algebras of a certain order n, using the unique factor decompositions of n (Theorem 4.2.7), theorem presented also in [19]). The question now was whether we could find a simpler way, simpler than already existing ones, to construct Wajsberg algebras of a certain order n. Using some existing results about MV-algebras (Proposition 4.1.8), we introduced a representation theorem for Wajsberg algebras (Theorem 4.2.8), presented also in [17]. The need to construct these algebras came, among other things, from the need to obtain examples of commutative and bounded BCK algebras, by means of which error-correcting codes could be transferred to MV algebras. In [17], we introduced such a method of constructing Wajsberg algebras of a certain order n (Remark 4.3.1) and, as an example, we constructed all Wajsberg algebras of order n, with $n \leq 9$. We have thus shown that for the order n, with n prime, we have a single Wajsberg algebra, an algebra that is totally ordered (up to an isomorphism). In particular, for orders 2, 3, 5, 7, we have a totally ordered Wajsberg algebra (up to an isomorphism). For orders 4 (Section 4.4.1), 6, and 9, we have 2 Wajsberg algebras, one totally ordered and one partially ordered, up to an isomorphism. As for all Wajsberg algebras of order 6, they are 25 (Section 4.4.2). For order 9, we have 5040 Wajsberg algebras (Section 4.4.4). For order 8, we have three Wajsberg algebras, up to an isomorphism, one totally ordered and two partially ordered. In their totality, they are 1441 Wajsberg algebras of order 8 (Section 4.4.3).

1.2.3 Algorithm for constructing commutative bounded BCK-algebras

Starting from the Wajsberg algebras construction algorithm (Remark 4.3.1), introduced in [17], and from the equivalence relation between Wajsberg algebras and bounded commutative BCK-algebras, we introduced in [18] a construction algorithm for bounded commutative BCK-algebras (see Section 4.5). Also, we presented some examples of such algebras obtained with this algorithm (see Section 4.6).

1.2.4 New ways to build BCK-algebras

During the research on codes attached to logical algebras, it became increasingly clear that we needed as many examples of BCK-algebras as possible. The existing literature offers few such examples, the subject of the construction of BCK-algebras being a very actual one and of great interest. Among the bibliographic sources with examples of BCK algebras we mention: [34], which presents BCK-algebras up to order 5, constructed using the axiomatic system of BCK-algebras, [13], in which BCI-algebras up to order 4 are constructed using the existing order relation on such an algebra, and [27], in which a 16th order BCK-algebra is constructed, starting from a 2nd order BCK-algebra using the variables existing in the neutrosophic logic system. Another known way to obtain examples of BCI or BCK-algebras is by using extensions from n-order algebras to $n+1$-order algebras. In the case of BCK-algebras, the only known extension is the Iseki extension, presented in Section 3.2.4. The problem with this extension is that the obtained algebras lose the commutativity property, in case when the source algebra had this property. So, we wanted not only to obtain examples of BCK-algebras, but also to obtain examples of BCK-algebras that have certain properties. A first result is the introduction of two new extensions for BCK-algebras, the Pseudo-Iseki extension (Section 3.3.1) and the involution extension (Section 3.3.3), results also presented in [42]. By pseudo-Iseki extension, higher order algebras with the properties of the source algebra can be obtained, less the property to be bounded. On the other hand, by extension through involution, we obtain bounded and commutative BCK-algebras of order $n+1$, from commutative BCK-algebras of order n. In this case, however, the property of being positive implicative of an BCK-algebra disappears in some cases. A second result was presented in [43] and consists in the introduction of a generalized process of extension for BCK-algebras, called the "generalized Iseki construction" for BCK-algebras. At the same time, we have shown that the Iseki extension and the pseudo-Iseki extension are particular cases of this process (see Section 3.3.2). Also in [43], starting from the example in [27], we analyzed in detail the behavior of the elements of a positive implicative BCK-algebra and we introduced new valid relations for this type of BCK-algebras (Theorem 3.4.1). Based on this theorem, we constructed a positive implicative BCK-algebra of order 6 (Proposition 3.4.6) and one of order 14 (Proposition 3.4.8).

1.2.5 BCK-trees

Starting from the construction of BCK-algebras, using their definition, we noticed that there are three 3rd order BCK-algebras, up to an isomorphism. In fact, they are 5, the first 4 being isomorphic two by two (see Section 3.2.1). Using this observation, but also the fact that any BCK-algebra of order n contains a BCK-algebra of order $n-1$, we obtained interesting results regarding the structure of the class of BCK-algebras. One such result is that any n-order BCK-algebra contains a 3rd-order BCK-algebra (Proposition 3.5.3) and that we can therefore construct n-order algebras by successive extensions from the 3rd order to the nth order. The structure thus created is a graph consisting of three trees, trees that we named BCK-trees (Definition 3.5.8). Interesting about these trees is that they each contain BCK-algebras with a certain property. At the same time, these trees are not independent, they have intersection points (Remark 3.5.9) represented by isomorphic BCK-algebras (Remark 3.5.10). At each such intersection point, that BCK-algebras lose one or more of their properties. These results were also presented in [42].

1.2.6 Computer algorithm to generate BCK-algebras

In order to obtain as many examples of BCK-algebras as possible, based on the previously mentioned results, we wrote a program that would generate BCK-algebras of a certain order. The program is written in the Python programming language ([45]), but can be easily transferred to any other programming language. The program generates BCK-algebras up to order 6, in a reasonable time (details can be found in Section 6.2). The algorithm starts from the three 3rd order BCK algebras (see Section 3.2.1) and generates the three BCK trees (see Section 3.5.2) up to the required order. The technique used is similar to the depth-first search technique used for graphs (see [47] Page 191). To generate a n-order BCK-algebra, the algorithm takes the table of an $n-1$ order BCK-algebra and borders it with a new row and a new column, thus obtaining a table with n rows and n columns. The ends of the line and column are fixed by the axioms of the definition of BCK-algebras. Thus, there remain $(n-2)$ positions on the new row and the new column which can be given values between 0 and $(n-1)$. Iteratively, the algorithm gives values to these positions, starting from 0 and going to $(n-1)$, obtaining at each step a new table with n rows and n columns. The algorithm verifies, at each assignment, not to give the value 0 at the same time to corresponding positions in the table, in order to respect axiom 4 from the definition of BCK-algebras. For each table thus generated, it is checked whether it meets axiom 1 from the definition of BCK-algebras, the other four being respected from the construction. If a BCK-algebra is identified, the algorithm also checks what its properties are. The program therefore proves its usefulness not only in generating examples, but also in verifying whether a certain table can represent a BCK-algebra or in verifying the properties of a BCK-algebra. Given the equivalences between BCK-algebras and other types of algebras, the program can be extended to work with other types of algebras and even perform conversions between different types of logical algebras. So far, we are not aware about the existence of a similar program. The complexity of such an algorithm was first discussed in [42], where we showed that the number of tables that must be checked to find BCK-algebras, when we make the transition from order $n-1$ to order n, is $(n^{(n-2)})^2-(2^{(n-2)}-1)$ (Proposition 3.5.1) and that the total number of tables to be checked for all orders from 2 to n is $\prod_{i=2}^{n}(i^{(i-2)})^2-(2^{(i-2)}-1)$ (Proposition 3.5.2). From these calculations, it was easy to see that we are dealing with a complexity of $O((n^{(n-2)})^2)$, which showed us that this activity is quite difficult even from small orders (see Section 3.5). However, the algorithm manages to generate, in a few seconds, BCK-algebras up to order 5. For order 6 it takes almost an hour. However, this is a step forward, because, apart from a few disparate examples, the literature did not contain examples of BCK-algebras larger than order 5. Through the resulting program, we obtained all the 6th order algebras, and even some of the 7th order. The algorithm in detail as well as the source code were presented in [43]. Being a first variant of such an algorithm, it is obvious that improvements can be made. A first improvement we made was to parallelize the generation process for BCK-trees. Thus, if a computer has 3 or more processor cores, each BCK-tree will be generated using one of these cores. In the sequential version, the first BCK-tree is generated first, then the second and finally the third. An improvement that could be made to this program is the introduction of a system for storing generated algebras. Thus, instead of generating all algebras up to the order n at each run, it could generate only the missing ones. Another improvement could be to change the programming language to a faster one, Python being recognized as a slow language. These improvements will be the subject of further researches.

Last but not least, the generation algorithm could be parallelized at the table level, so that the generation of the tables and even the verification of axiom 1, are done in parallel.

1.3 Work structure

After a chapter of preliminaries (Chapter 2), in which we acquaint the reader with the concepts used throughout the work, we begin the journey into the world of algebras with a first chapter (Chapter 3), about BCK-algebras. In this chapter, we will see how, starting from studies about the codes attached to BCK-algebras, new results were discovered about BCK-algebras and about the structure of this class of algebras. The chapter will present results from [42] and [43]. It is shown that the structure of the BCK-algebra class consists of 3 trees, each tree having algebras with a certain property (Section 3.5.2). These trees are not independent (Remark 3.5.9). They have intersection points. These points of intersection are represented by isomorphic BCK-algebras between them (Remark 3.5.10). Interestingly, at these intersection points, BCK-algebras lose properties. At the same time, new extensions for BCK-algebras (Section 3.2.3) and new identities valid for positive implicative BCK-algebras (Theorem 3.4.1) are introduced. The purpose of the chapter is to find new ways to construct higher-order BCK-algebras, starting from small order BCK-algebras. The establishment of these results was also based on the use of a computerized algorithm for generating BCK-algebras, an algorithm built by the author and presented in [43], included as a separate chapter in this work (see Chapter 6).

In the next chapter, (Chapter 4), new results on Wajsberg algebras are presented. An important place is occupied by the Wajsberg algebra representation theorem (Theorem 4.2.8) and the Wajsberg algebra construction algorithm based on this theorem (Remark 4.3.1). These results are also the product of studying binary codes attached to logical algebras. Also, in this chapter, using the equivalence between commutative and bounded BCK-algebras with Wajsberg algebras, we present an algorithm for constructing commutative and bounded BCK-algebras, this being another way of constructing BCK-algebras (Section 4.5). At the end of the chapter, as an example of the construction algorithm of Wajsberg algebras, the Wajsberg algebras of order n with $n \leq 9$, as well as the construction of bounded and commutative BCK-algebras starting from Wajsberg algebras are presented. The results presented are from [17], those related to Wajsberg algebras, as well as from [18], those related to commutative and bounded BCK-algebras.

In the Chapter 5, the codes attached to the BCK-algebras, based on [26] and [15], are presented as an introduction (Section 5.1). After, in the following sections, the codes attached to the Wajsberg algebras and MV-algebras, as well as results about the properties of these codes, are presented (Sections 5.2.1 and 5.2.2). These results were presented first in [19].

The last chapter (Chapter 6) presents in detail the computerized algorithm for generating BCK-algebras, as well as its properties. In this work, we have included an appendix, Appendix A, which presents all BCK-algebras of order n, with $n \leq 5$ from the third BCK-tree. Of course, the work could not be completed without a chapter of conclusions in which we will present our conclusions and various possibilities for further research on the subject.

Keywords: BCK-algebras, MV-algebras, Wajsberg algebras, BCK-trees, BCK extensions, computerised algorithm for BCK-algebras, binary block-codes.

1.4 Acknowledgements

This work is partialy supported by the project ANTREPRENORDOC, in the framework of Human Resources Development Operational Programme 2014-2020, financed from the European Social Fund under the contract number 36355/23.05.2019 HRD OP /380/6/13 – SMIS Code: 123847.

1.5 List of own publications

1. S. Hoskova-Mayerova C. Flaut, D. Flaut and **R. Vasile**, *From old ciphers to modern communications*, Advances in Military Technology **14** (2019), 79–88.

2. Arsham Borumand Saeid, Cristina Flaut, Sarka Hoskova-Mayerova and **Radu Vasile**, *Wajsberg algebras of order n, $n \leq 9$*, Neural Computing and Application **32** (2020), no. 17, 13301–13312.

3. Sarka Hoskova-Mayerova, Cristina Flaut and **Radu Vasile**, *Some remarks regarding finite bounded commutative BCK-algebras*, Algorithms as a foundation for modern applied Mathematics (Cristina Flaut, ed.), Studies in Fuzziness and Soft Computing, Springer, 2020.

4. Cristina Flaut and **Radu Vasile**, *Wajsberg algebras arising from binary block-codes*, Soft Computing **24** (2020), no. 8, 6047–6058.

5. **Radu Vasile**, *New ways for building BCK-algebras of higher-order*, Tbilisi Mathematical Journal **13** (2020), no. Special Issue iecmsa-2019, 111–124.

6. **Radu Vasile**, *New remarks about positive implicative BCK-algebras*, Italian journal of pure and applied mathematics (2021), accepted.

Bibliography

[1] M. Akram and Hee Sik Kim, *On K-algebras and BCI-algebras*, International Mathematical Forum **2** (2007), no. 12, 583–587.

[2] S.M. Alamouti, *A simple transmit diversity technique for wireless communications*, IEEE J. Selected Areas Communications (1998), no. 16, 1451–1458.

[3] I.C. Baianu, *Cell genome and interactome nonlinear dynamic models in Łukasiewicz logic algebras. neoplastic transformation models in a lukasiewicz topos.*, Cogprints preprint (2004), June 25th, GN 0406050.

[4] R. Belohlavek and V. Vilem Vychodil, *Residuated lattices of size* ≤ 12, Order **27** (2010), 147–161.

[5] Arsham Borumand Saeid, Cristina Flaut, Šárka Hošková-Mayerová, Morteza Afshar Jahanshahi, and Marjan Kuchaki Rafsanjani, *Some connections between BCK-algebras and n-ary block codes*, Soft Computing **22** (2018).

[6] D. Buşneag, *Categories of algebraic logic*, Editura Academiei Române, Bucureşti, 2006.

[7] C.C. Chang, *Algebraic analysis of many-valued logic*, Trans. Amer. Math. Soc. **88** (1958), 467–490.

[8] C.C. Chang, *A new proof of the completeness of the lukasiewicz*, Trans. Amer. Math. Soc. **93** (1959), 74–90.

[9] R Cignoli, I M. L. D'Ottaviano, and D Mundici, *Algebraic foundations of many-valued reasoning*, Trends in Logic, Studia Logica Library, vol. 7, Kluwer Academic Publishers, Dordrecht, 01 2000.

[10] R. Cignoli and A. T. Torell, *Boolean products of MV-algebras: Hypernormal MV-algebras*, J. Math. Anal. Appl. **199** (1996), 637–653.

[11] Steven T. Dougherty, *Algebraic coding theory over finite commutative rings*, Springer, 2017.

[12] W.A. Dudek, *On embedding hilbert algebras in BCK-algebras*, Mathematica Moravica **3** (1999), 25–28.

[13] Wieslaw Dudek, *Remarks on the axioms system for BCI-algebras*, Prace Naukowe WSP w Czestochowie, ser. Matematyka **2** (1996), 46–61.

[14] Li Fanzhang, Zhang Li, and Zhang Zhao, *Lie group machine learning*, de Gruyter, 2019.

[15] C. Flaut, *BCK-algebras arising from block-codes*, Journal of Intelligent & Fuzzy Systems **4** (2015), no. 28, 1829–1833.

[16] C. Flaut, D. Flaut, S. Hosková-Mayerová, and **R. Vasile**, *From old ciphers to modern communications*, Advances in Military Technology **14** (2019), 79–88.

[17] Cristina Flaut, Sárka Hosková-Mayerová, Arsham Borumand Saeid, and **Radu Vasile**, *Wajsberg algebras of order n, $n \leq 9$*, Neural Computing and Application **32** (2020), no. 17, 13301–13312.

[18] Cristina Flaut, Sárka Hosková-Mayerová, and **Radu Vasile**, *Some remarks regarding finite bounded commutative BCK-algebras*, Algorithms as a foundation for modern applied Mathematics (Cristina Flaut, ed.), Studies in Fuzziness and Soft Computing, Springer, 2020.

[19] Cristina Flaut and **Radu Vasile**, *Wajsberg algebras arising from binary block-codes*, Soft Computing **24** (2020), no. 8, 6047–6058.

[20] Josep Font, Antonio-Jesús Rodríguez, and Antoni Torrens, *Wajsberg algebras*, Stochastica **8** (1984), 5–31.

[21] Hernando Gaitan, *About quasivarieties of P-algebras and wajsberg algebras*, Ph.D. thesis, Iowa State University, 1990, Retrospective Theses and Dissertations. 9440; https://lib.dr.iastate.edu/rtd/9440.

[22] U. Hohle and S. E. Rodabaugh, *Mathematics of fuzzy sets: Logic, topology and measure theory*, Springer Science and Business Media, LLC, 1999.

[23] Min Hong, Sung, Jun, Young Bae, and Mehmet Öztürk, *Generalizations of BCK-algebras*, Scientiae Mathematicae Japonicae **58** (2004).

[24] Yasuyuki Imai and Kiyoshi Iséki, *On axiom systems of propositional calculi, xiv*, Proc. Japan. Academic. **42** (1966), 19–22.

[25] K. Iséki and S. Tanaka, *An introduction to the theory of BCK-algebras*, Math. Jpn. **23** (1978), 1–26.

[26] Y. B. Jun and S. Z. Song, *Codes based on BCK-algebras*, Inform. Sciences. (2011), no. 181, 5102–5109.

[27] Young Bae Jun, Seok-Zun Song, Florentin Smarandache, and Hashem Bordbar, *Neutrosophic quadruple BCK/BCI-algebras*, Axioms **7** (2018), 16.

[28] Hee Sik Kim and Young Hee Kim, *On BE-algebras*, Scientiae Mathematicae Japonicae **66** (2007).

[29] Kyung Kim and Yong Ho Yon, *Dual BCK-algebra and MV-algebra*, Scientiae Mathematicae Japonicae **66** (2007).

[30] S. Ling and C. Xing, *Coding theory: A first course*, Cambridge University Press, 2004.

[31] Antonio Maturo, Šárka Hošková-Mayerová, Daniela-Tatiana Soitu, and Janusz Kacprzyk (eds.), *Recent trends in social systems: Quantitative theories and quantitative models*, Studies in Systems, Decision and Control, vol. 66, ch. Some Connections Between Binary Block Codes and Hilbert Algebras, pp. 249 – 256, Springer International Publishing Switzerland, 2017.

[32] W McCulloch and W. Pitts, *A logical calculus of ideas immanent in nervous activity*, Bulletin of Mathematical Biophysics **5** (1943), 115–133.

[33] Alko R. Meijer, *Algebra for cryptologists*, Springer, 2016.

[34] J. Meng and Y. B. Jun, *BCK-algebras*, Kyung Moon Sa Co., Seoul, 1994.

[35] Sambasiva Rao Mukkamala, *A course in BE-algebras*, Springer Nature Singapore Pte Ltd., Singapore, 2018.

[36] D. Mundici, *MV-algebras-a short tutorial*, Department of Mathematics Ulisse Dini, University of Florence, 2007.

[37] Frédérique E. Oggier, Jean-Claude Belfiore, and Emanuele Viterbo, *Cyclic division algebras: A tool for space-time coding*, Foundations and Trends in Communications and Information Theory **4** (2007), no. 1, 1–95.

[38] S Pumplün and Thomas Unger, *Space-time block codes from nonassociative division algebras*, Computing Research Repository - CORR **5** (2010).

[39] A.B. Saeid, H. Fatemidokht, C. Flaut, and M. Kuchaki Rafsanjanib, *On codes based on BCK-algebras*, Journal of Intelligent & Fuzzy Systems **29** (2015), no. 5, 2133–2137.

[40] Zahra Samaei, Mohammad Azadani, and Leila Ranjbar, *A class of BCK-algebras*, Int. J. Algebra **5** (2011), no. 28, 1379–1385.

[41] Richard D. Schafer, *An introduction to nonassociative algebras*, Dover Publications, 2017.

[42] **Radu Vasile**, *New ways for building BCK-algebras of higher-order*, Tbilisi Mathematical Journal **13** (2020), no. Special Issue iecmsa-2019, 111–124.

[43] ______, *New remarks about positive implicative BCK-algebras*, Italian journal of pure and applied mathematics (2021), accepted.

[44] Thomas Unger and Nadya Markin, *Quadratic forms and space-time block codes from generalized quaternion and biquaternion algebras*, Computing Research Repository - CORR **57** (2008).

[45] Guido Van Rossum and Fred L. Drake, *Python 3 reference manual*, CreateSpace, Scotts Valley, CA, 2009.

[46] J. Wang, T. Davvaz, B. He, and P. F., *On derivations of MV-algebras*, 2010, https://arxiv.org/pdf/1709.04814.pdf.

[47] Ryan T. White and Archana Tikayat Ray, *Practical discrete mathematics - discover math principles that fuel algorithms for computer science and machine learning with python*, Packt Publishing Ltd, Livery Place 35 Livery Street Birmingham B3 2PB, UK, 2021.

[48] J. Zhan, B. Davvaz, W. A. Dudek, Y. B. Jun, and H. S. Kim, *Fuzzy logical algebras and their applications*, The Scientific World Journal (2015), Article ID 682648.