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Milnor algebras and projective hypersurfaces

PhD thesis
SUMMARY

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Abstract

Let f be a homogenous polynomial with complex coefficients. Let $M(f)$ denotes the corresponding Milnor algebra and $V(f)$ the hypersurface defined by the equation $f = 0$ in the complex projective space. The algebra $M(f)$ is a graded $\mathbb{C}-$algebra. The aim of this Thesis is to determine the Hilbert-Poincaré series of the Milnor algebra $M(f)$ in terms of the geometry of the hypersurface $V(f)$. The result is classically known for the case when $V(f)$ is smooth. The goal of this research is to discuss the case when $V(f)$ has only isolated singularities.

First we construct a free resolution for the Milnor (or Jacobian) algebra $M(f)$. In particular, this resolution implies that the dimensions of the graded components $M(f)_k$ are constant for $k \geq 2d-3$.

Then we show that the Milnor algebra of a nodal plane curve C has such a behaviour if and only if all the irreducible components of C are rational.

For the Chebyshev curves, all of these components are in addition smooth, hence they are lines or conics and explicit factorizations are given in this case.

We give sharp lower bounds for the degree of the syzygies involving the partial derivatives of a homogeneous polynomial defining a nodal hypersurface. The result gives information on the position of the singularities of a nodal hypersurface expressed in terms of defects or superabundances.

The case of Chebyshev hypersurfaces is considered as a test for this result and leads to a potentially infinite family of nodal hypersurfaces having nontrivial Alexander polynomials.

Keywords

Chebyshev and Fermat polynomials, projective hypersurfaces, singularities, nodal hypersurfaces, Milnor algebras, free resolutions, syzygies, Hilbert-Poincaré series.

CONTENTS

The aim of this thesis is to investigate some objects from Algebraic Geometry, namely the singular projective hypersurfaces, using primarily the technique of graded Milnor algebras and some related invariants. There is an important piece of data associated to any graded Milnor algebra, called the Hilbert-Poincaré series. For these series, we introduce new invariants to understand and quantify the difference between such a series and the corresponding series associated to a smooth hypersurface, say of Fermat type.

We compute and discuss Hilbert-Poincaré series associated to Milnor algebras for a lot of homogeneous polynomials (defining curves, surfaces, higher dimensional hypersurfaces, both the smooth and the singular cases).

It is natural to ask how these invariants depend on the different local singularity types (nodal, cuspidal and so on). One notices that the geometry can be very different in the smooth and the singular cases.

Let $S = \mathbb{C}[x_0, ..., x_n]$ be the graded ring of polynomials in $x_0, , ..., x_n$ with complex coefficients and denote by S_r the vector space of homogeneous polynomials in S of degree r. For any polynomial $f \in S_r$ we define the *Jacobian ideal* $J_f \subset S$ as the ideal spanned by the partial derivatives $f_0, ..., f_n$ of f with respect to $x_0, ..., x_n$. For $n = 2$ we use x, y, z instead of x_0, x_1, x_2 and f_x, f_y, f_z instead of f_0, f_1, f_2, in the same way as in Eisenbud's book [12].

The Hilbert-Poincaré series of a graded S-module M of finite type is defined by

$$HP(M)(t) = \sum_{k \geq 0} \dim M_k t^k$$

.

and it is known, see for instance [13], to be a rational function of the form

$$HP(M)(t) = \frac{P(M)(t)}{(1-t)^{n+1}}.$$

. For k sufficiently large, $\dim M_k = H(M)(k)$ is polynomial in k and $H(M)$ is called the Hilbert polynomial of M.

For any polynomial $f \in S_r$ we define the corresponding graded *Milnor* (or *Jacobian*) *algebra* by

$$M = M(f) = S/J_f.$$

.

In fact, such a Milnor algebra can be seen (up to a twist in grading) as the first (or the last) homology (or cohomology) of the Koszul complex of the partial derivatives $f_0, ..., f_n$ in S, see [5] or [6], Chapter 6, as well as our discussion at the beginning of Chapter 4.

The study of such Milnor algebras is related to the singularities of the corresponding projective hypersurface $D : f = 0$, see [5] and it is conveniently expressed by using the Hilbert-Poincaré series.

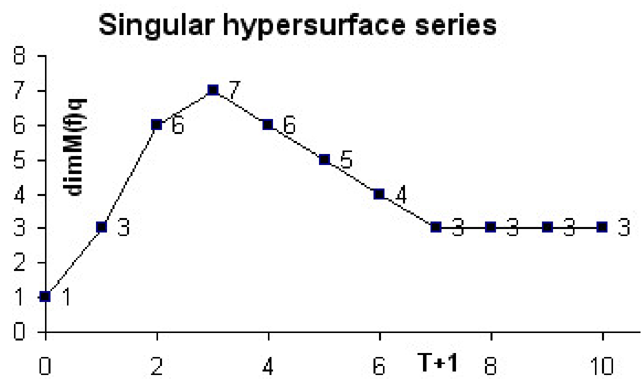
In other words, to determine the series $HP(M)(t)$ it is enough to determine the polynomial $P(M)(t)$. The most complete way to understand a Milnor algebra $M(f)$ is to construct a free resolution for $M(f)$. However, these are very difficult to obtain in general, the only case treated completely is the case of Chebyshev curves, see Chapter 3.

This thesis are based on join works [8], [9], [10] and derives mainly from the paper [5]. One of our research aims will be to improve the bounds in Choudary-Dimca Theorem from [5] to get sharp estimates in many cases.

Choudary-Dimca Theorem Let $V(f) : f = 0$ a hypersurface in $\mathbb{P}^n$ with only isolated singularities. For any $q \geq T+1, T = (n+1)(d-2)$, one has

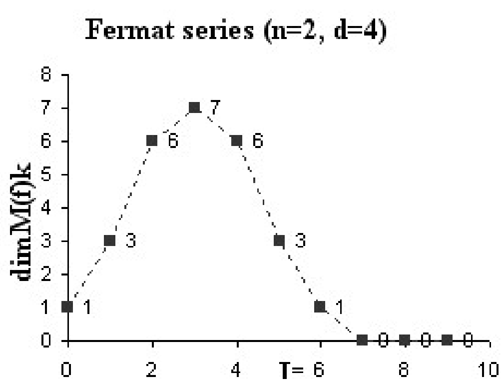
$$\dim M(f)_q = \tau(V(f)) = \sum_{j=1,p} \tau(V(f), a_j)$$

where $\tau(V(f))$ is the global Tjurina number of the hypersurface $V(f)$. In particular, the Hilbert polynomial $H(M(f))$ is constant and this constant is $\tau(V(f))$.

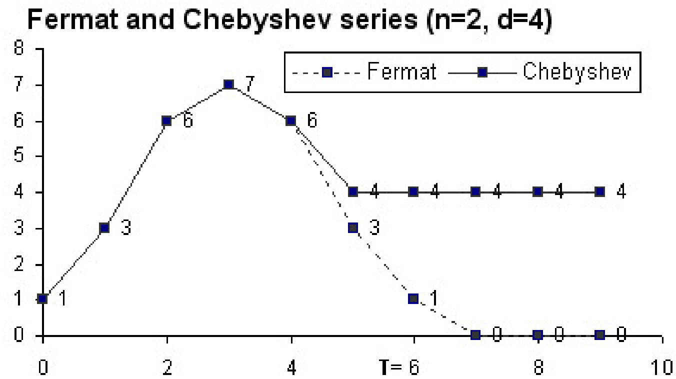


It gives the asymptotic behavious of the Hilbert Poincaré series $HP(M(f))$ for polynomials f defining a hypersurface with isolated singularities in the projective space $\mathbb{P}^n$.

If the hypersurface is smooth (say of Fermat type), the constant is zero and the Hilbert-Poincaré series is polynomial.



The behavior of Hilbert-Poincaré series for some singular hypersurfaces (see below Chebyshev hypersurfaces) are compared with smooth case, and here Choudary-Dimca Theorem is essential.



These examples suggest that the stabilization of the dimensions $\dim M(f)_q$ occurs earlier than predicted by Choudary-Dimca Theorem.

In general, we consider the following simplified approach, which it is more likely to work for large classes of singular hypersurfaces. For a hypersurface $D : f = 0$ in $\mathbb{P}^n$ with isolated singularities we introduce **four** integers, as follows:

Definition

(i) The *coincidence threshold* $ct(D)$ defined as

$$ct(D) = \max\{q \ : \ \dim M(f)_k = \dim M(f_s)_k \text{ for all } k \leq q\},$$

with f_s a homogeneous polynomial in S of degree $d = \deg f$ such that $D_s : f_s = 0$ is a smooth hypersurface in $\mathbb{P}^n$.

(ii) The *stability threshold* $st(D)$ defined as

$$st(D) = \min\{q \ : \ \dim M(f)_k = \tau(D) \text{ for all } k \geq q\}$$

where $\tau(D)$ is the total Tjurina number of D, i.e. the sum of all the Tjurina numbers of the singularities of D.

(iii) The *minimal degree of a nontrivial syzygy* $mdr(D)$ defined as

$$mdr(D) = \min\{q \ : \ H^n(K^*(f))_{q+n} \neq 0\}$$

where $K^*(f)$ is the Koszul complex of $f_0, ..., f_n$ with the natural grading defined in [9].

(iv) Let $D_{smooth} : f_{smooth} = 0$ be a smooth hypersurface of the same degree d in $\mathbb{P}^n$. We define the integer $def(D)$ =defect of D as

$$\text{def(D)} = \text{the first not zero coefficient of the difference } S(t) - F(t)$$

where $S(t)$ (resp. $F(t)$ are the corresponding Hilbert-Poincaré series of Milnor Algebras $M(f)$ (resp. $M(f_{smooth})$.

Recall also that, for a finite set of points $\mathcal{N} \subset \mathbb{P}^n$, we denote by

$$\operatorname{def} S_m(\mathcal{N}) = |\mathcal{N}| - \operatorname{codim}\{h \in S_m \mid h(a) = 0 \text{ for any } a \in \mathcal{N}\},$$

the *defect (or superabundance) of the linear system of polynomials in* S_m *vanishing at the points in* $\mathcal{N}$, see [6], p. 207. This positive integer is called the *failure of* $\mathcal{N}$ *to impose independent conditions on homogeneous polynomials of degree* m in [11].

Moreover it is easy to see that one has

$$ct(D) = mdr(D) + d - 2 \text{ and } S(t) - F(t) = \operatorname{def}(D)t^{ct(D)+1} + \text{ higher order terms}$$

with

$$def(D) = \operatorname{def} S_{(n+1)(d-2)-ct(D)-1}(\mathcal{N}).$$

Note that computing the Hilbert-Poincaré series of the Milnor algebra $M(f)$ using an appropriate software is much easier than computing the defects $\operatorname{def} S_k(\mathcal{N})$, because the Jacobian ideal comes with a given set of $(n+1)$ generators $f_0, ..., f_n$, while the ideal I of polynomials vanishing on $\mathcal{N}$ has not such a given generating set. However, it is the defects $\operatorname{def} S_k(\mathcal{N})$, who describe the position of the singularities of D in $\mathbb{P}^n$ and which occur in many geometric problems, giving information on the syzygies among $f_0, ..., f_n$.

The first main result in Chapter 3 gives such a resolution for the Milnor (or Jacobian) algebra $M(f)$ of a complex projective Chebyshev plane curve $C : f = T_d(x, y, z) = 0$ of degree d. This resolution depends on the parity of d and implies that the dimensions of the graded components $M(f)_k$ are constant for $k \geq 2d - 3$.

Conversely, we show that the Milnor algebra of a nodal plane curve C has such a behaviour if and only if all the irreducible components of C are rational.

For the Chebyshev curves, all of these components are in addition smooth, hence they are lines or conics and explicit factorizations are given in this case.

When the hypersurface $D : f = 0$ is smooth, then the partial derivatives $f_0, ..., f_n$ in S form a regular sequence in S and hence the only syzygies among them are the trivial ones, i.e. those in the module of Koszul relations $KR(f)$ defined at the beginning of Chapter 4. To study all such syzygies in general, it is convenient to work with the Koszul complex of the partial derivatives and to concentrate on the essential relations.

In Chapter 4, we discuss in detail the syzygies of nodal hypersurfaces. For instance we show that for a nodal curve there are no nontrivial relations

$$R_m : \ af_x + bf_y + cf_z = 0$$

with a, b, c homogeneous of degree $m < d - 2$ and we describe the number of independent relations of degree $m = d - 2$ in terms of the irreducible factors f_j of f.

We give sharp lower bounds for the degree of the syzygies involving the partial derivatives of a homogeneous polynomial defining a nodal hypersurface. The result gives information on the position of the singularities of a nodal hypersurface expressed in terms of defects or superabundances.

When D is a degree d nodal hypersurface in $\mathbb{P}^n$, with $\mathcal{N}$ as singular set, it follows that one has

$$\operatorname{def} S_k(\mathcal{N}) \neq 0 \text{ for } k < T - ct(D) \text{ and } \operatorname{def} S_k(\mathcal{N}) = 0 \text{ for } k \geq T - ct(D)$$

and also

$$\operatorname{def} S_k(\mathcal{N}) = |\mathcal{N}| - \dim S_k \text{ for } k \leq T - st(D)$$

where $T = (n+1)(d-2)$.

In Chapter 5 we consider the special case of Chebyshev hypersurfaces, which are classical examples of nodal hypersurfaces with many singularities. They were introduced by S. V. Chmutov to construct complex projective hypersurfaces with a large number of nodes, i.e. A_1-singularities, see [1], volume 2, p. 419 and [4].

For such hypersurfaces we compute the number of nodes and the stability threshhold $st(D)$, which is very hard to determine in general. We state as an open problem here, giving an explicit value for the coincidence threshhold $ct(D)$ of Chebyshev hypersurfaces. This conjecture is checked for a lot of cases in the Appendix B, but it seems to be very hard to prove theoretically.

In Chapter 6, we show a surprising relation between some topological invariants of singular projective hypersurfaces, namely the Alexander polynomials, and our algebraic invariants coming from the graded Milnor algebra $M(f)$.

The Alexander polynomials of singular hypersurfaces were introduced by A. Libgober [19], [20] and are very subtle invariants of the topology of the complement U. However the number of classes of hypersurfaces where these Alexander polynomials are not trivial is rather limited, and this explains the interest of our new (potentially infinite number of) examples.

The first result, was known since a long time. However, it is only due to our recent work that the implication of this Theorem became clear. It shows that on one hand the lower bounds on the syzygies degree obtained in the general case are best possible for curves and 3-dimensional Chebyshev hypersurfaces of degree ≤ 20 (and probably for all odd dimensional Chebyshev hypersurfaces, and on the other hand it gives some topological applications, by computing the Alexander polynomials of Chebyshev hypersurfaces of dimension 2 and 3 with degree $d \leq 20$.

We list in **Chapter 7** computations for some of the known hypersurfaces, with many configurations of singularities, not only nodes A_1. Many of our results were suggested by these computations, build in a databases, useful to extract new informations. Here we discuss more or less classical exemples of singular plane curves, surfaces in $\mathbb{P}^3$ with many nodes, and a number of higher dimensional hypersurfaces. The information collected here is likely to be very useful for future research in this area.

To the best of our knowledge, the only general formulas about Hilbert-Poincaré series of the Milnor algebra associated to smooth hypersurface is the following:

$$F(t) = HP(M(f_s)) = \frac{(1-t^{d-1})^{n+1}}{(1-t)^{n+1}} = (1+t+t^2+\ldots+t^{d-2})^{n+1}.$$

We can write $F(t)$ in the form: $F(t) = \sum_{k=0}^{k=T} a_k t^k$, where $T = (d-2)(n+1)$.

In **Appendix A**, we show explicit formulas for the coefficients a_k. We also discuss recent work by Huh [17] and show that we get some log-concave sequences in our setting as well.

In **Appendix B**, we collect the computations for Chebyshev hypersurfaces in $\mathbb{P}^n$, $n = 2, \ldots,$ with degree $d = 3, \ldots, 10$, which were used in relation with our Conjectures from Chapter 5 and 6.

To compute easily and make large scale simulations, we build our procedures, written in the Singular language and we present them in **Appendix C**.

In conclusion, **our new main results are in three directions:**

(A) The study of Hilbert-Poincaré series of the Milnor algebra $M(f)$ for the Chebyshev hypersurfaces: completely done for curves in $\mathbb{P}^2$ via the construction of a free resolution, and some results for $\mathbb{P}^n$ in particular a formula for the stabilization threshold $st(D)$.

(B) The study of the syzygies involving the partial derivatives of a homogeneous polynomial defining a nodal hypersurface: lower bounds for the essential syzygies and the relation with the defects of linear systems vanishing at the nodes. These lower bounds are sharp for odd dimensional nodal hypersurfaces, as shown by the example of Chebyshev hypersurfaces.

(C) The study of the relations between Alexander polynomial of a nodal hypersurface D and minimal degree of essential syzygies for D, in particular the relation with the defect $def(D)$ defined above.

Majority of these results has been allready published in the following articles: [8], [9], [10]:

• A.Dimca and G.Sticlaru, Chebyshev curves, free resolutions and rational curve arrangements, Math. Proc. Camb. Phil. Soc. (2012), doi:10.1017/S0305004112000138.

• A.Dimca and G.Sticlaru, Koszul complexes and pole order filtrations, (2011), arXiv:1108.3976.

• A.Dimca and G.Sticlaru, On the syzygies and Alexander polynomials of nodal hypersurfaces, Mathematische Nachrichten (2012), doi:10.1002/201100326.

The use of computer algebra systems is essential for the research done in relation to the main results of this thesis. It will become clear that without computer algebra systems like Singular, developed in Kaiserslautern University, [26] we could not have obtain the main results of this thesis at all.

Numerical experiments with the CoCoA package [25] **and the Singular package** [26] have played a key role in the completion of this work.

References

[1] Arnold, V.I., Gusein-Zade, S.M., Varchenko, A.N., Singularities of Differentiable Maps. vols 1/2, Monographs in Math., **82/83**, Birkhäuser, Basel (1985/1988) (document)

[2] M. Bizzarri, Rational Parameterization of Algebraic Curves, Bachelor Thesis, Pilsen, 2008

[3] Bruce, J. W.; Wall, C. T. C. (1979), "On the classification of cubic surfaces", The Journal of the London Mathematical Society, 19 (2), 245-256.

[4] S.V.Chmutov, Examples of projective surfaces with many singularities, J.Algebraic Geom. 1 (1992), 191–196. (document)

[5] A. D. R. Choudary, A. Dimca, Koszul complexes and hypersurface singularities, Proc. Amer. Math. Soc. 121 (1994), 1009-1016. (document)

[6] A. Dimca, Singularities and Topology of Hypersurfaces, Universitext, Springer-Verlag, 1992. (document)

[7] A. Dimca, Topics on Real and Complex Singularities, Vieweg Advanced Lecture in Mathematics, Friedr. Vieweg und Sohn, Braunschweig, 1987,242+xvii pp.

[8] A.Dimca and G.Sticlaru, Chebyshev curves, free resolutions and rational curve arrangements, Math. Proc. Camb. Phil. Soc. (2012), doi:10.1017/S0305004112000138. (document)

[9] A.Dimca and G.Sticlaru, Koszul complexes and pole order filtrations, arXiv:1108.3976. (document)

[10] A.Dimca and G.Sticlaru, On the syzygies and Alexander polynomials of nodal hypersurfaces, arXiv:1111.1533, to appear in Mathematische Nachrichten. (document)

[11] D. Eisenbud, M. Green, and J. Harris, Cayley-Bacharach theorems and conjectures, Bull. Amer. Math. Soc. 33 (1996), 295–324. (document)

[12] D. Eisenbud, The Geometry of Syzygies: A Second Course in Algebraic Geometry and Commutative Algebra, Graduate Texts in Mathematics, Vol. 229, Springer 2005. (document)

[13] G.-M. Greuel, G. Pfister, A Singular Introduction to Commutative Algebra (with contributions by O. Bachmann, C. Lossen, and H. Schnemann). Springer-Verlag, 2002 (second edition 2007). (document)

[14] Ph. Griffith, J. Harris, Principles of Algebraic Geometry. Wiley, New York (1978)

[15] Ph. Griffiths, On the period of certain rational integrals I, II, Ann. Math. 90(1969), 460-541.

[16] R. Hartshorne, Algebraic Geometry, GTM 52, Springer 1977.

[17] J. Huh, Milnor numbers of projective hypersurfaces and the chromatic polynomial of graphs, J. Amer. Math. Soc. 25 (2012), 907-927. (document)

[18] P.-V. Koseleff, D. Pecker, F. Rouillier, Computing Chebyshev knots diagrams, arXiv:1001.5192v2.

[19] A. Libgober, Alexander invariants of plane algebraic curves. Proc. Symp. Pure Math., **40**, Part 2, 135–144 (1983). (document)

[20] A. Libgober, Homotopy groups of the complements to singular hypersurfaces, II, Annals of Math., 139(1994), 117–144. (document)

[21] Milnor J., Singular points of complex hypersurfaces, Priceton Univ. Press., 1968.

[22] Carlos Rito, On the computation of singular plane curves and quartic surfaces arXiv:0906.3480v3, 31 May, 2010.

[23] R. Kloosterman, Cuspidal plane curves, syzygies and a bound on the MW-rank, arXiv:1107.2043v2.

[24] Y. Xu, Lagrange interpolation on Chebyshev points of two variables, J. Approx. Theory, 87 (1996), p. 220–238.

[25] CoCoA: a system for doing Computations in Commutative Algebra. Available at http://cocoa.dima.unige.it (document)

[26] Decker, W.; Greuel, G.-M.; Pfister, G.; Schönemann, H.: Singular 3-1-5 — A computer algebra system for polynomial computations.http://www.singular.uni-kl.de (2012). (document)

[27] Wall, C.T.C., Notes on the classification of singularities, Proc. London Math. Soc. 48 (1984) 461-513.

[28] http://www.oliverlabs.net
[29] http://enriques.mathematik.uni-mainz.de/docs/Eflaechen.shtml
[30] http://mathworld.wolfram.com/topics/AlgebraicSurfaces.html
[31] http://www.singsurf.org/parade/Cubics.html