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BINOMIAL IDEALS ASSOCIATED WITH GRAPHS

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Preface

The main subject of this thesis is the study of binomial ideals in polynomial rings arising from combinatorics.

By a simple graph G on the vertex set $[n]$ we mean an undirected graph G with no loops and no multiple edges. Villareal [**30**] introduced monomial edge ideals $I(G)$ associated with a simple graph G in the polynomial ring $K[x_1,\dots,x_n]$ in n variables over a field K. The monomial edge ideal $I(G)$ is generated by all the monomials x_ix_j where $i<j$ and $\{i,j\}$ is an edge of G. In a similar way, one may define the binomial edge ideal $J_G \subset S = K[x_1,\dots,x_n,y_1,\dots,y_n]$ associated with G as the ideal generated by all the binomials $f_{ij} = x_iy_j - x_jy_i$ where $i<j$ and $\{i,j\}$ is an edge of G.

Binomial edge ideals were first introduced in [**19**] and [**22**] independently at the same time. The authors in [**19**] obtained some nice results on Gröbner bases, primary decomposition and minimal prime ideals of binomial edge ideals. Afterwards, many other research papers approached the topic of binomial edge ideals. Much effort has been done for studying the Cohen-Macaulay property of a binomial edge ideal J_G [**14, 23, 31**], the syzygies of J_G and its regularity [**20, 25, 27, 16, 29, 32**]. Part of the interest in studying binomial edge ideals comes from that fact that they have some applications to algebraic statistics [**19, 13, 24, 28**].

In this thesis we follow two directions. The first one is the description of homological properties of some classes of binomial edge ideals, namely, those associated with complete bipartite graphs, cycles, and block graphs. The second direction concerns the study of a class of binomial ideals which are associated with scrolls. The thesis mainly consists of the original results obtained in the papers [**9**], [**7**] and [**8**].

Now, we present the structure of our thesis.

In Chapter 1 we survey the fundamental notions and results which are intensively used throughout the thesis. We especially follow the books [**11, 13, 18**]. We recall some basic definitions and known facts about monomial ideals and several concepts such as minimal free resolutions and primary decompositions. We give a short introduction to the main features of the Gröbner basis theory, including the Buchberger criterion and algorithm.

In the second part of this chapter, we survey the main properties of binomial edge ideals, which are the main topic of this thesis, and consider examples of binomial edge ideals associated to the line graph and the complete graph. This part is mainly based on the fundamental paper [**19**] where the binomial edge ideals were introduced. We study the graphs G for which the generators form a Gröbner basis with respect to the lexicographic order once we have a given vertex labeling of G and we present a combinatorial method for finding the reduced Gröbner basis for any binomial edge ideal. Finally, we recall how the minimal prime ideals of J_G can be obtained from certain subsets of vertices of G.

Chapter 2 is based on our paper [**9**]. In this chapter, extremal Betti numbers of some classes of binomial edge ideals are studied. We show that the binomial edge ideal J_G and its initial ideal $\mathrm{in}_<(J_G)$ with respect to the lexicographic order have the same extremal Betti numbers for complete bipartite graphs and cycles. This is a partial positive answer to the conjecture proposed in [**14**] which states that, for any graph G, J_G and $\mathrm{in}_<(J_G)$ have the same extremal Betti numbers. We use the advantages of known results on the resolution of J_G given in [**29**] and [**32**]. In the first step, we find a minimal generating set for the initial ideals for these graphs by using Theorem 1.4.11 which characterizes in terms of admissible paths the reduced Gröbner basis of J_G with respect to the lexicographic order in the ring S. Proposition 2.3.1 shows that $\mathrm{in}_<(J_G)$ has linear quotients if G is a complete bipartite graph. For monomial ideals with linear quotients, one may easily compute the Betti numbers. Therefore, we may calculate all the graded Betti numbers of $\mathrm{in}_<(J_G)$ for the complete bipartite graph; see Theorem 2.3.2. Then we show that $\operatorname{proj\,dim} \mathrm{in}_<(J_G) = \operatorname{proj\,dim} J_G$ and $\operatorname{reg} \mathrm{in}_<(J_G) = \operatorname{reg} J_G$, and, therefore, $\mathrm{in}_<(J_G)$ has a unique extremal Betti number like J_G. Finally, we show that the extremal Betti number of $\mathrm{in}_<(J_G)$ is equal to that of J_G; see Corollary 2.3.3. While for the complete bipartite graph, drawing the desired conclusion was not so difficult, for cycles we need a bit more technique. In the first step, as in the complete bipartite case, we identify the minimal monomial generators of $\mathrm{in}_<(J_G)$ where G is an n–cycle with a natural labeling of its vertices. In this case, we use an induction argument (Lemma 2.4.1 and Lemma 2.4.3) to compute the projective dimension and the regularity of $\mathrm{in}_<(J_G)$. Finally, in Theorem 2.4.6, we show that J_G and $\mathrm{in}_<(J_G)$ have the same extremal Betti number.

In Chapter 3 we study binomial edge ideals of block graphs. This chapter is based on our joint paper [**8**]. By a block graph G we mean a chordal graph with the property that it is chordal and any two maximal cliques intersect in at most one vertex. In support of the conjecture in [**14**], we show, in Theorem 3.1.2, that, for a block graph G, $\operatorname{depth}(S/J_G) = \operatorname{depth}(S/\operatorname{in}_<(J_G)) = n+c$, where c is the number of connected components of G. We show a similar equality for regularity, namely $\operatorname{reg}(S/J_G) = \operatorname{reg}(S/\operatorname{in}_<(J_G)) = \ell$ if G is a C_ℓ-graph where ℓ is the length of the longest induced path of G. C_ℓ-graphs constitute a subclass of the block graphs. In [**20**], Matsuda and Murai showed that, for any connected graph G, we have $\ell \leq \operatorname{reg}(S/J_G) \leq n-1$. Therefore, we conclude that C_ℓ-graphs have minimal regularity.

The main motivation of our work was to answer the following question. May we characterize the connected graphs G whose longest induced path has length ℓ and $\operatorname{reg}(S/J_G) = \ell$? In other words, may we characterize the graphs whose binomial edge ideal has minimal regularity? We succeeded to answer this question for trees. We show that if T is a tree whose longest induced path has length ℓ, then $\operatorname{reg}(S/J_T) = \ell$ if and only if T is caterpillar; see Theorem 3.2.1.

In [**21**], the so-called weakly closed graphs were introduced. This is a class of graphs which includes closed graphs. In the same paper, it was shown that a tree is caterpillar if and only if it is a weakly closed graph. Having in mind our Theorem 3.2.1 and [**16**, Theorem 3.2] which states that $\operatorname{reg}(S/J_G) = \ell$ if G is a connected closed graph whose longest induced path has length ℓ, and by some computer experiments, we are tempted to formulate the following.

Conjecture. *If G is a connected weakly closed graph whose longest induced path has length ℓ, then* $\operatorname{reg}(S/J_G) = \ell$.

In Chapter 4, based on our joint paper [**7**], in analogy to the binomial edge ideal J_G generated by the 2-minors $f_{ij} = x_i y_j - x_j y_i$ of the matrix

$$X = \begin{pmatrix} x_1 & \dots & x_n \\ y_1 & \dots & y_n \end{pmatrix},$$

where $i < j$ and $\{i, j\}$ is an edge of G, we introduce the binomial edge ideal associated with the $2 \times n$ Hankel matrix

$$X = \begin{pmatrix} x_1 & \dots & x_{n-1} & x_n \\ x_2 & \dots & x_n & x_{n+1} \end{pmatrix}.$$

It is known that all the 2-minors of the Hankel matrix generate the ideal I_X of the rational normal curve $\mathscr{X} \subset \mathbb{P}^n$.

In Section 4.1, we show that the generators of I_G form a Gröbner basis with respect to the reverse lexicographic order if and only if G is closed with the given labeling. As a consequence of this theorem, we derive that, for a closed graph G, the ideal I_G is Cohen-Macaulay of dimension $1+c$, where c is the number of connected components of G.

In Section 4.2 we study the properties of I_G for a closed graph G. We compute the minimal prime ideals of I_G in Theorem 4.2.2 for a connected closed graph G. By using this theorem, we characterize those connected closed graphs G for which I_G is a radical ideal. In addition, we show that I_G is a set-theoretic complete intersection if G is connected and closed. We end by giving a sharp upper bound for the regularity of I_G and showing that I_G has a linear resolution if and only if G is a complete graph.

CHAPTER 1

Preliminaries

In this chapter we recall some fundamental notions and results that are used throughout the thesis.

1.1. Gröbner bases

1.1.1. Short survey on monomial ideals and their basic properties. Let K be a field and $S = K[x_1,..,x_n]$ be the ring of polynomials in n variables with coefficients in K. Let $\mathbb{Z}_+^n$ denote the set of vectors $a = (a_1,\dots,a_n) \in \mathbb{Z}^n$, $a_i \geq 0$, $i \in \{1,\dots,n\}$. As usual, we denote the set of non-negative integers by $\mathbb{N}$.

An element in S of the form $x_1^{a_1}\cdots x_n^{a_n}$ is called a monomial. We may represent a monomial u by $u = x^a$ where $a = (a_1,\dots,a_n) \in \mathbb{Z}_+^n$. Let $\mathrm{Mon}(S)$ be the set of all monomials in S. Any polynomial f in S can be expressed uniquely as a K-linear combination of monomials in $\mathrm{Mon}(S)$

$$f = \sum_{u \in \mathrm{Mon}(S)} a_u u, \text{ where } a_u \in K.$$

We call the set $\mathrm{supp}(f) = \{u \in \mathrm{Mon}(S) : a_u \neq 0\}$ the *support* of f.

If $u = x_1^{a_1}\cdots x_n^{a_n}$ is a monomial in S, one defines the *degree* of u as $\deg(u) = a_1 + \cdots + a_n$. If $f \in S \setminus \{0\}$ is a polynomial, the degree of f is $\deg f = \max\{\deg u : u \in \mathrm{supp}(f)\}$.

The ring S has an $\mathbb{N}$-grading given by $S = \oplus_{d\in\mathbb{N}} S_d$ where S_d is the K-vector subspace of S generated by all the monomials of degree d. A non zero element in S_d is called a *homogeneous polynomial* of degree d.

A *monomial ideal* $I \subset S$ is an ideal which is generated by a set of monomials. By Dickson's Lemma [**13**, Theorem 1.3], we know that any monomial ideal may be generated by a finite set of monomials. The following theorem explains an important property of monomial ideals.

THEOREM 1.1.1. [**13**] *Let I be a monomial ideal. The set $\mathscr{M}$ of monomials belonging to I is a K-basis of I.*

COROLLARY 1.1.2. [**13**] Let $I \subset S$ be an ideal. The following conditions are equivalent:

(i) I is a monomial ideal.
(ii) For every polynomial $f \in S$ we have that $f \in I$ if and only if $\operatorname{supp}(f) \subset I$.

COROLLARY 1.1.3. [**13**] Let $I \subset S$ be a monomial ideal. The residue classes of the monomials not belonging to I form a K-basis of the ring S/I.

EXAMPLE 1.1.4. Let $I = (x_1^{a_1}, \dots, x_n^{a_n}) \subset S$. Then a K-basis of S/I is given by the residue classes of all monomials $w = x_1^{b_1} \cdots x_n^{b_n} \in S$ with the property that $b_i < a_i$ for all $1 \le i \le n$. Therefore, we have $\dim_K(S/I) = a_1 \cdots a_n$.

PROPOSITION 1.1.5. [**13**] Let a set of monomials $\{u_1, \dots, u_m\}$ form a set of generators for the monomial ideal I. Then the monomial v belongs to I if and only if there exists a monomial w such that $v = wu_i$ for some i.

PROPOSITION 1.1.6. [**13**] Let $I \subset S$ be a monomial ideal and let $G(I)$ denote the set of monomials in I which are minimal with respect to divisibility. Then $G(I)$ is the unique minimal set of monomial generators of I.

Obviously, the polynomial ring S is $\mathbb{Z}^n$-graded with graded components

$$S_a = \begin{cases} Kx^a, & \text{if } a \in \mathbb{Z}_+^n, \\ 0, & \text{otherwise.} \end{cases}$$

Let $f = cx^a \in S$ with $c \in K$ and $a \in \mathbb{Z}^n$. Then f is called homogeneous of degree a.

We observe that any monomial ideal $I \subset S$ is a $\mathbb{Z}^n$-graded submodule of S. In this case, the quotient S/I is also $\mathbb{Z}^n$-graded. In other words,

$$I = \oplus_{x^a \in I} S_a \text{ and } S/I = \oplus_{x^a \notin I} S_a.$$

1.1.1.1. *Standard algebraic operations on monomial ideals.* Let I and J be some ideals in S. The sum and the product of the ideals are defined as:

$I + J = \{f + g : f \in I, g \in J\}$ and $IJ = (G)$, where $G = \{fg : f \in I, g \in J\}$.

Obviously, $I+J$ and IJ are again monomial ideals if I and J are monomial ideals. In addition, we have $G(I+J) \subset G(I) \cup G(J)$ and $G(IJ) \subset G(I)G(J)$.

The intersection of two monomial ideals I and J is also a monomial ideal which is given by

$$I \cap J = (\{\text{lcm}(u,w) : u \in G(I), w \in G(J)\}).$$

The ideal quotient of two monomial ideals is also a monomial ideal which is given by

$$I : J = \bigcap_{w \in G(J)} I : (w),$$

where

$$I : (w) = (\{u/\gcd(u,w) : u \in G(I)\}).$$

The *radical* of a monomial ideal I is the ideal given by

$$\sqrt{I} = (\sqrt{u} : u \in G(I)),$$

where, for $u = x^a$, $\sqrt{u} = \prod_{i=1, a_i \neq 0}^{n} x_i$. For example, if $u = x_1^3 x_2 x_4^2$, then $\sqrt{u} = x_1 x_2 x_4$.

I is called a radical ideal if $\sqrt{I} = I$. Note that a monomial ideal I is a radical ideal if and only I is a square free monomial ideal, that is, the minimal monomial generators are squarefree monomials.

EXAMPLE 1.1.7. Let $I = (x^3, x^2y, y^3)$ and $J = (xy, y^2)$ be monomial ideals in the polynomial ring $S = K[x,y]$. Then

$$I + J = (x^3, x^2y, y^3) + (xy, y^2) = (x^3, x^2y, y^3, xy, y^2) = (x^3, xy, y^2).$$

Since xy divides x^2y and y^2 divides y^3, we may remove the generators x^2y and y^3. The product of I and J is

$$IJ = (x^3, x^2y, y^3)(xy, y^2) = (x^4y, x^3y^2, x^2y^3, xy^4, y^5),$$

and the intersection is

$$\begin{aligned} I \cap J &= (\text{lcm}(x^3, xy), \text{lcm}(x^3, y^2), \text{lcm}(x^2y, xy), \ldots, \text{lcm}(y^3, y^2)) \\ &= (x^3y, x^3y^2, x^2y, x^2y^2, xy^3, y^3) \\ &= (x^2y, y^3). \end{aligned}$$

Finally, the quotient of the two ideals is

$$\begin{aligned} I:J &= (x^3/\gcd(x^3,xy), x^2y/\gcd(x^2y,xy), y^3/\gcd(y^3,xy)) \\ &\cap (x^3/\gcd(x^3,y^2), x^2y/\gcd(x^2y,y^2), y^3/\gcd(y^3,y^2)) \\ &= (x^2,x,y^2) \cap (x^3,x^2,y) \\ &= (x,y^2) \cap (x^2,y) \\ &= (x^2,xy,y^2). \end{aligned}$$

1.1.2. Short survey on Gröbner basis theory. In the polynomial algebra $K[x]$ with one variable over a field K, we use long division for given polynomials $f,g \in K[x]$ with $g \neq 0$. There exist uniquely determined polynomials q and r in $K[x]$ such that $f = qg + r$ where deg $r <$ deg g.

The algorithm to calculate q and r is as follows: If deg $f <$ deg g then we set $q = 0$ and $r = f$. If deg $f \geq$ deg g, we calculate $r_1 = f - (a/b)x^{n-m}$, where ax^n and bx^m are the leading terms of f and g, respectively. If deg $r_1 <$ deg g, then $q = (a/b)x^{n-m}$ and $r = r_1$. Otherwise, we do the same reduction to r_1. The algorithm terminates in finitely many steps.

The theory of Gröbner bases is based on the generalization of this algorithm to polynomial algebras with several variables. In this case, we encounter a problem which is about determining leading terms and comparing monomials containing more than one variable. To fix this problem, we are going to present monomial orders.

1.1.2.1. *Monomial orders.* We call the pair $(X,\leq)$ a partially ordered set if X is a set and $\leq$ is a binary relation on X which is reflexive, antisymmetric and transitive, i.e. for all a,b, and c in X we have:

(i) $a \in X \Rightarrow a \leq a$;
(ii) $a \leq b,\ b \leq a \Rightarrow a = b$;
(iii) $a \leq b,\ b \leq c \Rightarrow a \leq c$.

We write $a < b$ to mean $a \leq b$ and $a \neq b$. Also, $a \geq b$ is the same as $b \leq a$.

EXAMPLE 1.1.8.

(1) The set of all subsets of X, the power set of X, is denoted by $\mathscr{P}(X)$. The inclusion relation $\subseteq$ is a partial order on $\mathscr{P}(X)$.
(2) The binary relation $|$ on monomials in Mon(S) is defined as follows:

$$x_1^{a_1}\cdots x_n^{a_n} | x_1^{b_1}\cdots x_n^{b_n} \text{ if } a_1 \leq b_1,\ldots,a_n \leq b_n.$$

In this case, we say that $x_1^{a_1}\cdots x_n^{a_n}$ divides $x_1^{b_1}\cdots x_n^{b_n}$. We can check easily that the set (Mon(S),$|$) is a poset.

A partial order $\leq$ on X is called a *total order*, if for any two elements $a,b \in X$ we have $a \leq b$ or $a \geq b$. In other words, all pairs of elements of X are comparable with respect to $\leq$.

We define a total order on the set of all monomials in $S = K[x_1,\ldots,x_n]$ which respects the multiplicative structure on Mon(S).

DEFINITION 1.1.9. A *monomial order* on S is a total order $\leq$ on Mon(S) which satisfies:

(i) $1 \leq u$, for all $u \in$ Mon(S);
(ii) if $u \leq v$, then for every $w \in$ Mon(S), $uw \leq vw$.

Let us underline that every two monomials can be compared with respect to a monomial order. The following conditions are satisfied for any monomial order.

PROPOSITION 1.1.10. [**13**] Let $\leq$ be a monomial order on S. Then, the following hold:

(i) if $u,v \in$ Mon(S) with $u|v$, then $u \leq v$;
(ii) if $u_1,u_2,\ldots$ is a sequence of monomials with $u_1 \geq u_2 \geq \ldots$ then there exists an integer m such that $u_i = u_m$ for all $i \geq m$.

We now present some standard monomial orders on S. In these examples we denote the ordering of the variables in a standard way as $x_1 > x_2 > \ldots > x_n$. Let x^a and x^b be two monomials in S.

- The *lexicographic order*: We have $x^a < x^b$, if either $\sum_{i=1}^n a_i < \sum_{i=1}^n b_i$ or $\sum_{i=1}^n a_i = \sum_{i=1}^n b_i$ and the leftmost non zero component of the vector $a-b$ is negative. In this ordering we first compare total degrees, and next we compare the powers of the variables starting with the lowest indexed variable.

  EXAMPLE 1.1.11. $x_1^2x_2x_4^2x_5^3 < x_1^2x_2x_4^4x_5$, since the two monomials have the same degree and we have $a-b = (0,0,0,-2,2)$.

- The *pure lexicographic order*: We have $x^a < x^b$ if the leftmost non zero component of the vector $a-b$ is negative. In this order total degree is not important.

  EXAMPLE 1.1.12. $x_1^3x_2x_4^5 < x_1^3x_2x_3$, because we have $a-b = (0,0,-1,5)$.

- The *reverse lexicographic order*: We have $x^a < x^b$ if either $\sum_{i=1}^n a_i < \sum_{i=1}^n b_i$ or $\sum_{i=1}^n a_i = \sum_{i=1}^n b_i$ and the rightmost non zero component of the vector $a-b$ is positive.

  EXAMPLE 1.1.13. $x_1^2x_3x_4 < x_1x_2^2x_4$, because we have $a-b = (1,-2,1,0)$.

The difference between the lexicographic and the reverse lexicographic order could be explained in the following way. Let $x^a, x^b \in \mathrm{Mon}(S)$ be two monomials of same degree. If $x^a < x^b$ in the lexicographic order, x^b has more from the beginning than x^a. If $x^a < x^b$ in the reverse lexicographic order, x^b has less from the end than x^a.

EXAMPLE 1.1.14. Consider all monomials in $S = K[x_1, x_2, x_3]$ of degree 2. In the lexicographic order: $x_1^2 > x_1x_2 > x_1x_3 > x_2^2 > x_2x_3 > x_3^2$. In the reverse lexicographic order: $x_1^2 > x_1x_2 > x_2^2 > x_1x_3 > x_2x_3 > x_3^2$.

1.1.2.2. *Initial ideals and Gröbner bases.* Let $<$ be a fixed monomial order on the polynomial ring $S = K[x_1, \ldots, x_n]$ over a field K. For a non zero polynomial $f \in S$ the *initial monomial* of f with respect to $<$ is the biggest monomial among the monomials belonging to $\mathrm{supp}(f)$. The initial monomial of f is denoted by $\mathrm{in}_<(f)$ with respect to $<$. The *leading coefficient* $c \in K$ of f is the coefficient of $\mathrm{in}_<(f)$ and the *leading term* of f is $c\,\mathrm{in}_<(f)$.

EXAMPLE 1.1.15. Let $f = 5x_1^3x_2^2x_3 + x_1^2x_2^4 + 3x_1^4x_3$. If $<$ is the lexicographic order, then $\mathrm{in}_<(f) = x_1^3x_2^2x_3$; if $<$ is the reverse lexicographic order, then $\mathrm{in}_<(f) = x_1^2x_2^4$, and if $<$ is the pure lexicographic order, then $\mathrm{in}_<(f) = x_1^4x_3$.

Initial monomials of the sum and the product of two polynomials are given by the following lemma:

LEMMA 1.1.16. [**13**] Let f and g be nonzero polynomials and $<$ be a monomial order on S. Then

(i) $\mathrm{in}_<(fg)=\mathrm{in}_<(f)\mathrm{in}_<(g)$;
(ii) $\mathrm{in}_<(f+g) \leq \max\{\mathrm{in}_<(f), \mathrm{in}_<(g)\}$. Equality holds if $\mathrm{in}_<(f) \neq \mathrm{in}_<(g)$.

Let $I \subset S$ be a nonzero ideal. The *initial ideal* of I is a monomial ideal which is generated by all the initial monomials of the nonzero polynomials belonging to I. The initial ideal of I with respect to a monomial order $<$ is denoted by $\mathrm{in}_<(I)$. Thus, $\mathrm{in}_<(I) = (\mathrm{in}_<(f) : f \in I, f \neq 0)$.

Note that $\mathrm{in}_<(I) = (0)$ if $I = (0)$. In general, the initial monomials of the elements of a generating set do not generate $\mathrm{in}_<(I)$. For example, consider the ideal $I = (x_1^2 - x_1x_2 + x_2, x_1 - x_2)$ and the pure lexicographic order in the polynomial ring $K[x_1, x_2]$. We have $(x_1^2 - x_1x_2 + x_2) - x_1(x_1 - x_2) = x_2 \in I$. If we assume that $\mathrm{in}_<(I) = (x_1^2, x_1) = (x_1)$, then we should have $x_2 \in \mathrm{in}_<(I)$. However $x_2 \notin (x_1)$.

According to Proposition 1.1.6, a monomial ideal has a unique minimal set of monomial generators. Dickson's Lemma says that this minimal generating set is a finite set. Therefore every monomial ideal is finitely generated. Since $\text{in}_<(I)$ is a monomial ideal, there exist $g_1,\ldots,g_m \in I$ such that $\text{in}_<(I) = (\text{in}_<(g_1),\ldots,\text{in}_<(g_m))$.

DEFINITION 1.1.17. Let $I \subset S$ be a nonzero ideal and let $<$ be a monomial order on S. A set of polynomials $\{g_1,\ldots,g_m\}$ is said to be a *Gröbner basis* of I with respect to the order $<$ if $\text{in}_<(I) = (\text{in}_<(g_1),\ldots,\text{in}_<(g_m))$.

According to the definition, for any nonzero monomial ideal there always exists a Gröbner basis.

EXAMPLE 1.1.18. The set $G = (x_1^2 - x_1x_2 + x_2, x_1 - x_2)$ is not a Gröbner basis for $I = (G)$ with respect to the pure lexicographic order, since, as it is explained before, $x_2 \in I$. However $x_2 \notin (x_1^2, x_1) = (x_1)$.

THEOREM 1.1.19. [**13**] *Let I be an ideal on S and let the set $\{g_1,\ldots,g_m\}$ be a Gröbner basis of I with respect to a monomial order $<$. Then, $I = (g_1,\ldots,g_m)$. In other words, every Gröbner basis of I is a generating set for I.*

COROLLARY 1.1.20 (Hilbert's basis theorem). [**13**] Every ideal in the polynomial ring $S = K[x_1,\ldots,x_n]$ is finitely generated. In other words, the ring S is Noetherian.

THEOREM 1.1.21 (The division algorithm). [**13**] *Let f and $g_1,\ldots,g_m$ be nonzero polynomials in S and let $<$ be a monomial order. There exist polynomials r and $q_1,\ldots,q_m$ in S with $f = q_1g_1 + \ldots + q_mg_m + r$ such that the following conditions are satisfied:*

(i) no element of $\text{supp}(r)$ *is contained in* $(\text{in}_<(g_1),\ldots,\text{in}_<(g_m))$*;*
(ii) $\text{in}_<(f) \geq \text{in}_<(q_ig_i)$ *for all i.*

The expression $q_1g_1 + \ldots + q_mg_m + r$ satisfying the conditions above is called a *standard expression* for f. The polynomial r is a *remainder* of f with respect to $g_1,\ldots,g_m$. The following example shows that the standard expression of f is not unique.

EXAMPLE 1.1.22. The polynomial $f = x_1^2 - x_2^3$ has two different standard expressions with respect to $g_1 = x_1 + x_2$ and $g_2 = x_1 + x_2^2$. We consider the pure lexicographic order. We have $f = x_1g_1 - x_2g_2$ and $f = (x_1 - x_2)g_1 + x_2^2 - x_2^3$. In these different standard expressions we have different remainders: 0 and $x_2^2 - x_2^3$.

If f has a zero remainder with respect to polynomials $g_1,\ldots,g_m$, then we say that f *reduces to* 0 with respect to $g_1,\ldots,g_m$.

We now describe an algorithm to find a standard expression for f with respect to an ordered set of polynomials $g_1,\ldots,g_m$. In this algorithm we obtain a finite sequence of polynomials $h_i, 1 \leq i \leq s$, in the following way:

We let $h_0 = f$ and assume that we have already defined the polynomials $h_1, \ldots, h_i$. The sequence ends with h_i if the polynomial h_i satisfies $\mathrm{supp}(h_i) \notin (\mathrm{in}_<(g_1), \ldots, \mathrm{in}_<(g_m))$.

Otherwise let u be the biggest monomial in $\mathrm{supp}(h_i)$ which belongs to $(\mathrm{in}_<(g_1), \ldots, \mathrm{in}_<(g_m))$ and let j be the smallest integer such that $(\mathrm{in}_<(g_j) \mid u$.

We define $h_{i+1} = h_i - ab^{-1}wg_j$, where $w = u/\mathrm{in}_<(g_j)$ and a and b are the leading coefficients of h_i and g_j, respectively. Suppose that the sequence of h_i's ends with h_s. Then we obtain the following equations:

(1) $f = h_0 = q'_1 g_{j_1} + h_1$
(2) $h_1 = q'_2 g_{j_2} + h_2$
(3) $h_2 = q'_3 g_{j_3} + h_3$
$\vdots$
(4) $h_{s-1} = q'_s g_{j_s} + h_s$

Replacing h_1 in (1) by (2), we obtain $f = q'_1 g_{j_1} + q'_2 g_{j_2} + h_2$. In this new expression, instead of h_2 we write the expression (3). By repeating this process, we obtain a standard expression for f with the remainder $r = h_s$.

EXAMPLE 1.1.23. Let $f = x_1^2x_2 + x_1x_2^2 - 3x_2^3$. We calculate a standard expression for f with respect to $g_1 = x_1 - x_2$ and $g_2 = x_2$ by using the algorithm described above for the lexicographic order.

$$f = h_0 = x_1x_2g_1 + 2x_1x_2^2 - 3x_2^3 \qquad \text{where } h_1 = 2x_1x_2^2 - 3x_2^3$$
$$h_1 = 2x_2^2g_1 - x_2^3 \qquad \text{where } h_2 = -x_2^3$$
$$h_2 = -x_2^2g_2 \qquad \text{where } h_3 = 0.$$

Therefore the standard expression is $f = (x_1x_2 + 2x_2^2)g_1 - x_2^2g_2$. f reduces to 0 with respect to $x_1 - x_2$ and x_2.

PROPOSITION 1.1.24. [**13**] Let $<$ be a monomial order on S and the set $\{g_1, \ldots, g_m\}$ be a Gröbner basis for the ideal $I = (g_1, \ldots, g_m)$. Then, any nonzero polynomial f in S has a unique remainder with respect to $g_1, \ldots, g_m$.

COROLLARY 1.1.25. [**13**] Let the set $\{g_1, \ldots, g_m\}$ be a Gröbner basis for the ideal $I = (g_1, \ldots, g_m)$. Then, any polynomial $f \in S$ which belongs to I reduces to zero with respect to $g_1, \ldots, g_m$.

We present an algorithm that constructs a Gröbner basis of an ideal from any given set of generators. We need the following definition.

DEFINITION 1.1.26. Let f and g be two polynomials on S and let $<$ be a monomial order. The polynomial

$$S(f,g) = \frac{\mathrm{lcm}(\mathrm{in}_<(f), \mathrm{in}_<(g))}{c\,\mathrm{in}_<(f)} f - \frac{\mathrm{lcm}(\mathrm{in}_<(f), \mathrm{in}_<(g))}{d\,\mathrm{in}_<(g)} g$$

is called the *S-polynomial* of f and g with respect to $<$.

Recall that $\operatorname{lcm}(\mathrm{in}_<(f),\mathrm{in}_<(g))$ stands for the least common multiple of $\mathrm{in}_<(f)$ and $\mathrm{in}_<(g)$. In the formula, c and d are the leading coefficients of f and g, respectively.

EXAMPLE 1.1.27. Let $f = x_1^3x_2 + x_1x_2 + x_3^2$ and $g = 2x_1^2 + x_2x_3$. Then $\operatorname{lcm}(\mathrm{in}_<(f),\mathrm{in}_<(g)) = x_1^3x_2$ with respect to the lexicographic order, and therefore the S-polynomial of f and g is

$$S(f,g) = \frac{x_1^3x_2}{x_1^3x_2}(x_1^3x_2 + x_1x_2 + x_3^2) - \frac{x_1^3x_2}{2x_1^2}(2x_1^2 + x_2x_3) = -1/2x_1x_2^2x_3 + x_1x_2 + x_3^2.$$

Notice that the S-polynomial helps us cancel the leading terms of f and g and obtain another polynomial in the same ideal with different leading term.

The next theorem gives us a method to check for a given ideal $I = (g_1,\ldots,g_m)$ if the generating set $\{g_1,\ldots,g_m\}$ forms a Gröbner basis for I.

THEOREM 1.1.28 (Buchberger's criterion). **[13]** *Let $I = (g_1,\ldots,g_m)$ be an ideal of S and $<$ a monomial order on S. Then $G = \{g_1,\ldots,g_m\}$ is a Gröbner basis of I with respect to $<$ if and only if $S(g_i,g_j)$ reduces to zero with respect to G for all $i < j$.*

To calculate S-polynomials of $I = (g_1,\ldots,g_m)$ for all pairs of generators can be cumbersome. The following proposition helps us avoid the calculations in some cases.

PROPOSITION 1.1.29. **[13]** Let f and g be polynomials in S with a monomial order $<$. If initial monomials $\mathrm{in}_<(f)$ and $\mathrm{in}_<(g)$ are relatively prime then $S(f,g)$ reduces to 0 with respect to f and g.

EXAMPLE 1.1.30. Let $I \subset K[x_1,\ldots,x_n,y_1,\ldots,y_n]$ be generated by all two minors of the matrix

$$\begin{pmatrix} x_1 & x_2 & \ldots & x_n \\ y_1 & y_2 & \ldots & y_n \end{pmatrix}.$$

In other words, $I = (\{f_{ij} = x_iy_j - x_jy_i : 1 \le i < j \le n\})$. Let $<$ be the lexicographic order in $K[x_1,\ldots,x_n,y_1,\ldots,y_n]$ induced by $x_1 > x_2 > \ldots > x_n > y_1 > \ldots > y_n$. We want to show that all S-polynomials $S(f_{ij}, f_{kl})$ have a remainder zero, where $\{i, j\} \neq \{k,l\}$. If $i \neq k$ and $j \neq l$, then $\mathrm{in}_<(f_{ij})$ and $\mathrm{in}_<(f_{kl})$ are relatively prime, so $S(f_{ij}, f_{kl})$ has a remainder 0. If $i = k$, we may assume that $j < l$ and we have $S(f_{ij}, f_{kl}) = S(f_{ij}, f_{il}) = -x_jy_iy_l + x_ly_iy_j = -y_i f_{jl}$ which is a standard expression of $S(f_{ij}, f_{kl})$ with a remainder 0.

If $j = l$, we may assume that $i < k$ and we get $S(f_{ij}, f_{kl}) = S(f_{ij}, f_{kj}) = x_i x_j y_k - x_k x_j y_i = x_j f_{ik}$, which is again a standard expression with remainder 0. Consequently, the set of minors $\{f_{ij} : 1 \leq i < j \leq n\}$ form a Gröbner basis of I with respect to the lexicographic order.

There exists an algorithm which allows us to compute a Gröbner basis for an ideal I by using a given set of generators for I. The algorithm named as **Buchberger's algorithm** is in fact a consequence of Theorem 1.1.28. Buchberger's algorithm works as follows:

Step 1: We compute the S-polynomial for each pair of elements of the generating set G of the ideal I.

Step 2: If all S-polynomials reduce to zero then G is a Gröbner basis of I. Otherwise we add one of the nonzero remainders to our system of generators to form a new system of generators and go back to Step 1.

Since any strictly ascending sequence of monomial ideals in S is finite, this algorithm ends after a finite number of steps.

EXAMPLE 1.1.31. Let $I = (x_1^2 + 2x_1x_2^2, x_1x_2 + 2x_2^3 - 1) \subset \mathbb{Q}[x_1, x_2]$. By using Buchberger's algorithm, we form a Gröbner basis for I in $S = K[x_1, x_2]$ with respect to the lexicographic order.

Let $f = x_1^2 + 2x_1x_2^2$, $g = x_1x_2 + 2x_2^3 - 1$. We compute the S-polynomial $S(f,g) = x_1$ of f and g. Since $S(f,g) = x_1 \notin (\mathrm{in}_<(f), \mathrm{in}_<(g))$, we add $h = x_1$ to the set of generators getting the new generating set $\{f,g,h\}$.

Now let us choose the pair g,h. Since the initial monomial of the S-polynomial $S(g,h) = 2x_2^3 - 1$ is not in $(\mathrm{in}_<(f), \mathrm{in}_<(g), \mathrm{in}_<(h))$, we get another generator, which is $t = 2x_2^3 - 1$ and the generating set becomes $\{f,g,h,t\}$.

Here we do not have to compute the S-polynomial of every pair, since we know that $S(f,g) = h$ and $S(g,h) = t$. We have all the other remainders equal to 0 as well:

$$
\begin{aligned}
S(f,h) &= 2x_1x_2^2 = 2x_2^2h \\
S(f,t) &= 1/2x_1^2 + 2x_1x_2^5 = 1/2f + x_1x_2^2t \\
S(g,t) &= 1/2x_1 + 2x_2^5 - x_2^2 = 1/2h + x_2^2t \\
S(h,t) &= 1/2x_1 = 1/2h
\end{aligned}
$$

It follows that the Gröbner basis is $\{x_1^2 + 2x_1x_2^2, x_1x_2 + 2x_2^3 - 1, x_1, 2x_2^3 - 1\}$.

We may add some more polynomials to the set G and still have a Gröbner basis for the ideal. However, under some conditions there is a unique Gröbner basis.

DEFINITION 1.1.32. A set $G = \{g_1, \ldots, g_m\}$ is a *reduced Gröbner basis* for $I \subset S$ with respect to a monomial order $<$ if G is a Gröbner basis for I with the following conditions satisfied:

(i) The leading coefficient of each g_i is 1;
(ii) For all $i \neq j$, no $u \in \mathrm{supp}(g_j)$ is divisible by $\mathrm{in}_<(g_i)$.

EXAMPLE 1.1.33. The reduced Gröbner basis of I in the previous example is $\{x_1, x_2^3 - 1/2\}$.

1.2. Minimal graded free resolutions of graded ideals

In this section, we present numerical data arising from the minimal graded free resolution of a quotient of a polynomial ring by a graded ideal.

Let us set, for this section, $S = K[x_1, \ldots, x_n]$ be a polynomial ring in n variables over a field K. Every graded ideal $I \subset S$ has a (unique up to isomorphism) *minimal graded free resolution*

$$F_\bullet : 0 \to F_p \to \cdots \to F_1 \to F_0 = S \to S/I \to 0,$$

where $F_i = \bigoplus_{j \in \mathbb{Z}} S(-j)^{\beta_{ij}}$, for any value of i. The exponents $\beta_{ij} = \beta_{ij}(S/I)$ are called the *graded Betti numbers* of S/I. The *total Betti numbers* of S/I are $\beta_i = \sum_j \beta_{ij}, i \geq 0$. The *projective dimension* of S/I is given by

$$\mathrm{proj\,dim}(S/I) = \max\{i : \beta_{ij} \neq 0, \text{ for some } j \in \mathbb{Z}\}.$$

According to the Auslander-Buchsbaum formula [**18**, Corollary A 4.3] we have

$$\mathrm{depth}\, S/I = n - \mathrm{proj\,dim}\, S/I.$$

We recall that $\mathrm{depth}\, S/I$ is the length of a maximal S/I–sequence of homogeneous elements contained in the maximal graded ideal of S. It is known that $\mathrm{depth}(S/I) \leq \dim(S/I)$. If the equality holds, we say that I is a Cohen-Macaulay ideal. Hence, I is Cohen-Macaulay if and only if $\mathrm{depth}(S/I) = \dim(S/I)$.

The *regularity* of S/I is defined as $\mathrm{reg}(S/I) = \max\{j - i : \beta_{ij} \neq 0\}$. All numerical data arising from the minimal graded free resolution of S/I are called the homological invariants of S/I.

Usually, the graded Betti numbers are displayed in the so-called *Betti diagram* of S/I which has the shape indicated in Figure 1.1.

The Betti numbers marked in the figure by fat dots are called *extremal Betti numbers*.

EXAMPLE 1.2.1. Let $J \subset S = K[x_1, .., x_5, y_1, .., y_5]$ be the ideal

$$J = (x_1y_2 - x_2y_1, x_2y_3 - x_3y_2, x_2y_4 - x_4y_2, x_3y_4 - x_4y_3, x_2y_5 - x_5y_2, x_4y_5 - x_5y_4)$$

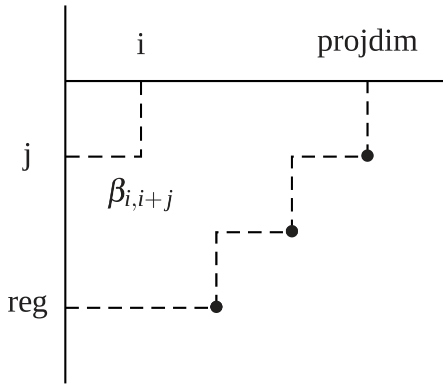


FIGURE 1.1

and its initial ideal $\mathrm{in}_<(J) \subset K[x_1,..,x_5,y_1,..,y_5]$, where $<$ is the lexicographic order induced by $x_1 > \cdots > x_5 > y_1 > \cdots > y_5$. The Betti diagrams of S/J and $S/\mathrm{in}_<(J)$ are displayed below.

According to the diagrams, $S/\mathrm{in}_<(J)$ and S/J have the same extremal Betti number which has the value of 4. We also have $\mathrm{proj\,dim}(S/J) = \mathrm{proj\,dim}(S/\mathrm{in}_<(J)) = 4$, hence $\mathrm{depth}(S/J) = \mathrm{depth}(S/\mathrm{in}_<(J)) = 6$ and $\mathrm{reg}(S/J) = \mathrm{reg}(S/\mathrm{in}_<(J)) = 2$.

J :

	0	1	2	3	4
0	1	–	–	–	–
1	–	6	4	–	–
2	–	–	9	12	4
Total	1	6	13	12	4

$\mathrm{in}_<(J)$:

	0	1	2	3	4
0	1	–	–	–	–
1	–	6	5	–	–
2	–	1	10	12	4
Total	1	7	15	12	4

The Betti diagrams help us to write down the minimal graded free resolution for each ideal:

$$0 \to S(-6)^4 \to S(-5)^{12} \to S(-3)^4 \oplus S(-4)^9 \to S(-2)^6 \to S \to S/J \to 0,$$

$$0 \to S(-6)^4 \to S(-5)^{12} \to S(-3)^5 \oplus S(-4)^{10} \to$$

$$\to S(-2)^6 \oplus S(-3) \to S \to S/\mathrm{in}_<(J) \to 0.$$

DEFINITION 1.2.2. A graded ring S/I has a $(d-1)$-*linear resolution* (or, equivalently, I has a d-linear resolution) if its minimal graded free resolution is of the form

$0 \to S(-p)^{\beta_{p-d+1}} \to S(-p+1)^{\beta_{p-d}} \to \cdots \to S(-d-1)^{\beta_2} \to S(-d)^{\beta_1} \to S \to S/I \to 0.$ (*)

The definition says that S/I has a $(d-1)$-linear resolution if and only if $\beta_{ij}(S/I) = 0$ for $j \neq i+d-1$. In the Betti diagram of S/I, except for the position $\beta_{(0,0)} = 1$, all the other non-zero graded Betti numbers are located on the $(d-1)$st row. In other words, S/I has a $(d-1)$-linear resolution if and only if I is generated in degree d and $\operatorname{reg}(S/I) = d-1$.

EXAMPLE 1.2.3. Let $I \subset K[x_1,\ldots,x_4,y_1,\ldots,y_4]$, $I = (x_1y_2-x_2y_1, x_1y_3-x_3y_1, x_1y_4-x_4y_1, x_2y_3-x_3y_2, x_2y_4-x_4y_2, x_3y_4-x_4y_3)$. S/I has a 2–linear resolution. The Betti diagram of S/I is the following.

	0	1	2	3
0	1	–	–	–
1	–	6	8	3

REMARK 1.2.4. Let us make another comment on ideals with linear resolution. If S/I has a linear resolution then, by applying the additivity property of the Hilbert series in (*), we get

$$H_{S/I}(t) = \frac{1-\beta_1 t^d + \beta_2 t^{d+1} - \cdots + (-1)^{p-d+1}\beta_{p-d+1}t^p}{(1-t)^n}.$$

This formula shows that if S/I has a linear resolution, then the Betti numbers are determined by its Hilbert series $H_{S/I}$.

A comparison between the graded Betti numbers of S/I and $S/\operatorname{in}_<(I)$ where $<$ is a monomial order in S, is given in the following theorem and corollary.

THEOREM 1.2.5. [**18**] *For all i and j,* $\beta_{ij}(S/I) \leq \beta_{ij}(S/\operatorname{in}_<(I))$.

This inequality between the graded Betti numbers yields the following consequences.

COROLLARY 1.2.6. [**18**]

(i) $\operatorname{proj\,dim} S/I \leq \operatorname{proj\,dim}(S/\operatorname{in}_<(I))$.
(ii) $\operatorname{depth} S/I \geq \operatorname{depth}(S/\operatorname{in}_<(I))$.
(iii) $\operatorname{reg} S/I \leq \operatorname{reg}(S/\operatorname{in}_<(I))$.
(iv) If $S/\operatorname{in}_<(I)$ is Cohen-Macaulay then S/I is Cohen-Macaulay.
(v) If $S/\operatorname{in}_<(I)$ has a linear resolution then S/I has a linear resolution.

1.3. A brief review of primary decomposition

In this subsection we mainly follow the book [**11**, Chapter 3]. Let R be a ring and M an R–module. The prime ideal P of R is called an *associated prime* of M if there exists some $m \in M$ such that

$$P = (0 :_R m) = \{r \in R : rm = 0\}.$$

In particular, if $I \subset R$ is an ideal, then P is an associated prime of R/I (or, simply, of I) if $P = I : (a)$ for some $a \in R$.

We collect the main results on the associated primes of a module in the following theorem. We first recall that the set of all associated primes of M is usually denoted by $\mathrm{Ass}(M)$. Very often, we write $\mathrm{Ass}(I)$ instead of $\mathrm{Ass}(R/I)$ for an ideal I of R.

THEOREM 1.3.1. [**11**] *Let R be a Noetherian ring and M a finitely generated non-zero R–module. Then:*

(a) $\mathrm{Ass}(M)$ *is a non-empty set which contains the set of minimal prime ideals over the annihilator of M,* $\mathrm{Ann}(M)$, *where* $\mathrm{Ann}(M) = \{r \in R : rM = 0\}$. *In particular,* $\mathrm{Ass}(I) \supseteq \mathrm{Min}(I)$. *Here,* $\mathrm{Min}(I)$ *denotes the set of minimal prime ideals of I.*
(b) *We have $Z(M) = \bigcup_{P \in \mathrm{Ass}(M)} P$ where $Z(M)$ denotes the set of all the zero-divisors on M.*
(c) $\mathrm{Ass}(M)$ *commutes with localization. More precisely, if $S \subset R$ is a multiplicative set, then*

$$\mathrm{Ass}_{S^{-1}R}(S^{-1}M) = \{S^{-1}P : P \in \mathrm{Ass}(M), P \cap S = \emptyset\}.$$

(d) *If* $0 \to M' \to M \to M'' \to 0$ *is an exact sequence of R–modules, then*

$$\mathrm{Ass}(M') \subset \mathrm{Ass}(M) \subset \mathrm{Ass}(M') \cup \mathrm{Ass}(M'').$$

DEFINITION 1.3.2. Let R be a Noetherian ring and P a prime ideal of R. The ideal I of R is called a *P–primary ideal* (or simply a *primary ideal*) if $\mathrm{Ass}(R/I) = \{P\}$.

THEOREM 1.3.3. [**11**] *Let R be a Noetherian ring. Then any ideal I of R has a decomposition $I = Q_1 \cap \cdots \cap Q_r$, where:*

(a) *Q_i is P_i–primary for every i;*
(b) *the decomposition is irredundant, that is, no Q_i can be omitted in the decomposition;*
(c) *P_i are pairwise distinct.*

Moreover, $\mathrm{Ass}(I) = \{P_1, \ldots, P_r\}$.

The above decomposition is called a *primary decomposition* of I. We now survey the main results on the primary decomposition of monomial ideals, mainly following [**18**].

THEOREM 1.3.4. [**18**] *Let $I \subset S = K[x_1,\ldots,x_n]$ be a monomial ideal. Then $I = \bigcap_{i=1}^m Q_i$, where each Q_i is generated by pure powers of the variables. In other words, each Q_i is of the form $(x_{i_1}^{a_1},\ldots,x_{i_k}^{a_k})$.*

Moreover, it can be shown that the irredundant presentation constructed in the above proof is unique.

A monomial ideal is called *irreducible* if it can not be written as proper intersection of two other monomial ideals. It is called *reducible* if it is not irreducible.

COROLLARY 1.3.5. [**18**] A monomial ideal is irreducible if and only if it is generated by pure powers of the variables.

It follows from Theorem 1.3.4 and Corollary 1.3.5 that each monomial has a unique presentation as an irredundant intersection of irreducible monomial ideals, moreover, the proof of Theorem 1.3.4 gives us a procedure for finding such a presentation.

EXAMPLE 1.3.6. Let $I = (x_1^2x_2^3, x_2^2x_3, x_3^2)$. Then

$$\begin{aligned} I &= (x_1^2, x_2^2x_3, x_3^2) \cap (x_2^3, x_2^2x_3, x_3^2) \\ &= (x_1^2, x_2^2, x_3^2) \cap (x_1^2, x_3, x_3^2) \cap (x_2^3, x_2^2, x_3^2) \cap (x_2^3, x_3, x_3^2) \\ &= (x_1^2, x_2^2, x_3^2) \cap (x_1^2, x_3) \cap (x_2^3, x_3). \end{aligned}$$

For squarefree monomial ideals we have the following corollary.

COROLLARY 1.3.7. [**18**] Let $I \subset S$ be a squarefree monomial ideal. Then

$$I = \bigcap_{P \in \mathrm{Min}(I)} P,$$

and each $P \in \mathrm{Min}(I)$ is a monomial prime ideal.

Here $\mathrm{Min}(I)$ denotes, as usual, the set of minimal prime ideals of I.

We end this section by recalling that the primary decomposition obtained from an irredundant intersection of irreducible ideals is unique and we call it the *standard primary decomposition of* I.

1.4. Binomial edge ideals

Let G be a simple graph on the vertex set $[n]$ and with the edge set $E(G)$. We consider $S = K[x_1,\ldots,x_n,y_1,\ldots,y_n]$ to be the polynomial ring in $2n$ variables over a field K.

We define the binomial edge ideal $J_G \subset S$ associated with G as the ideal generated by all the binomials $f_{ij} = x_i y_j - x_j y_i$ where $1 \le i < j \le n$ with $\{i, j\} \in E(G)$.

Note that if G has an isolated vertex i, and G' is the restriction of G to the vertex set $[n] \setminus \{i\}$, then $J_G = J_{G'}$. For this reason, we will always assume that G has no isolated vertex.

We consider the polynomial ring S endowed with the lexicographic order induced by the natural order of variables $x_1 > x_2 > \cdots > x_n > y_1 > y_2 > \cdots > y_n$. We denote by $\mathrm{in}_<(J_G)$ the initial ideal of J_G with respect to this monomial order. The ideal $\mathrm{in}_<(J_G)$ is a monomial ideal minimally generated by the initial monomials of the binomials in the reduced Gröbner basis of J_G with respect to the lexicographic order.

EXAMPLE 1.4.1. In Figure 1.2, G is a simple graph on the vertex set $[6]$.

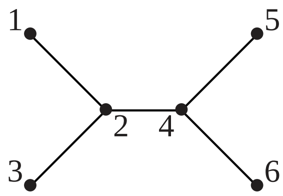


FIGURE 1.2

The binomial edge ideal of G is $J_G = (f_{12}, f_{23}, f_{24}, f_{45}, f_{46})$. The reduced Gröbner basis of J_G with respect to the lexicographic order is $\mathscr{G} = \{x_1y_2 - x_2y_1, x_2y_3 - x_3y_2, x_2y_4 - x_4y_2, x_4y_5 - x_5y_4, x_4y_6 - x_6y_4, x_3y_2y_4 - x_4y_2y_3, x_5y_4y_6 - x_6y_4y_5\}$. Therefore, the initial ideal of J_G is

$$\mathrm{in}_<(J_G) = (x_1y_2, x_2y_3, x_2y_4, x_4y_5, x_4y_6, x_3y_2y_4, x_5y_4y_6).$$

1.4.1. Binomial edge ideals with quadratic Gröbner bases. In this subsection, we are going to present two basic examples of binomial edge ideals. Both examples are ideals with the property that their generators form their reduced Gröbner bases.

The graphs G whose associated binomial edge ideal J_G shares the above property, in other words, J_G has a quadratic Gröbner basis, are described in combinatorial terms in the following theorem.

THEOREM 1.4.2. [**19**, Theorem 1.1] *Let G be a simple graph on the vertex set $[n]$ with the edge set $E(G)$, and let $<$ be the lexicographic order on S induced by $x_1 > \cdots > x_n > y_1 > \cdots > y_n$. Then the following conditions are equivalent:*

(a) *The generators f_{ij} of J_G form a quadratic Gröbner basis.*

(b) *For all edges* $\{i, j\}$ *and* $\{i,k\}$ *with* $j > i < k$ *or* $j < i > k$ *one has* $\{j,k\} \in E(G)$.

In other words, if we represent the edge $\{i, j\}$ with $i < j$ by an arrow which points from i to j, then we have the following picture for a graph which satisfies the condition (b) in Theorem 1.4.2.

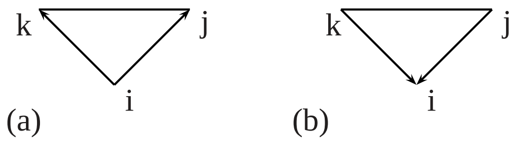


FIGURE 1.3

EXAMPLE 1.4.3. Let G be the graph with edges $\{1,2\}$ and $\{1,3\}$. We have $J_G = (x_1y_2 - x_2y_1, x_1y_3 - x_3y_1)$. We calculate the $S-$polynomial of f_{12} and f_{13}. We get $S(f_{12}, f_{13}) = y_1(x_2y_3 - x_3y_2) \in J_G$. Thus $S(f_{12}, f_{13}) = x_2y_1y_3 - x_3y_1y_2 \in J_G$.

But the initial monomial $x_2y_1y_3$ of $S(f_{12}, f_{13})$ does not belong to the ideal generated by the initial monomials of f_{12} and f_{13} which shows that $\{f_{12}, f_{13}\}$ is not a Gröbner basis of J_G.

However, for the same graph with the different labeling $\{1,2\},\{2,3\}$ the generators of J_G form a Gröbner Basis.

For the associated binomial edge ideal $J_G = (x_1y_2 - x_2y_1, x_2y_3 - x_3y_2)$, the $S-$polynomial of f_{12} and f_{23} reduces to 0 since the initial monomials of f_{12} and f_{23} are relatively prime.

DEFINITION 1.4.4. A graph G endowed with a labeling which satisfies condition (b) in Theorem 1.4.2 is called *closed with respect to the given labeling.*

Therefore, we may reformulate Theorem 1.4.2 by saying that the generators of J_G form a Gröbner basis with respect to the lexicographic order if and only if G is closed with respect to its given labeling.

We have showed that the graph from Figure 1.3(a), without edge $\{j,k\}$, is not closed for the labeling $i = 1, j = 2, k = 3$ and closed for the labeling $k = 1, i = 2, j = 3$.

DEFINITION 1.4.5. A graph G is *closed* if there exists a labeling of its vertices such that G is closed with respect to it.

The following graphs are examples of graphs which are not closed. Note that if a graph G is closed, then any induced subgraph of G should be closed as well. Thus, if a graph G contains as an induced subgraph any of the graphs in the example below then it is not closed.

EXAMPLE 1.4.6.

(i) The graph with three different edges e_1, e_2, e_3 such that $e_1 \cap e_2 \cap e_3 \neq \emptyset$ is called the *claw graph*. The claw graph is not closed; see Figure 1.4. Hence, any closed graph must be claw free.

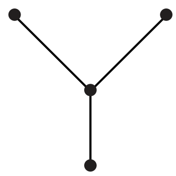


FIGURE 1.4

(ii) Any cycle C_n of length $n \geq 4$ is not closed. Assume that there exists a labeling of its vertices, $a_1, \ldots, a_n$ (labeled clockwise). To obtain a closed labeling, we should either choose $a_1 < a_2 < \cdots < a_n < a_1$ or $a_1 > a_2 > \cdots a_n > a_1$. Since C_n has no chord, both choices lead to contradiction.

For instance in C_4, if we label in clockwise direction, $1, 2, 3, 4$, then, $\{2, 4\}$ should belong to the edge set, since $\{1, 2\} \in E(C_4)$ and $\{1, 4\} \in E(C_4)$ and also $\{1, 3\}$ must belong to the edge set to have a closed graph since $\{1, 4\} \in E(C_4)$ and $\{3, 4\} \in E(C_4)$.

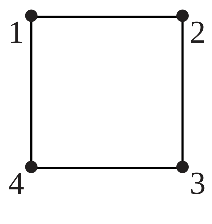


FIGURE 1.5

After this observation, let us state the following proposition.

PROPOSITION 1.4.7. If G is closed then G is a chordal graph, that is, it has no induced cycle of length ≥ 4 and it is claw-free.

One may also show that a bipartite graph is not closed unless it is a line graph; see [**19**, Corollary 1.3].

In the sequel, we study binomial edge ideals of two classes of closed graphs.

1.4.1.1. *The binomial edge ideal of the complete graph.* Let $G = K_n$ be the complete graph on the vertex set $[n]$. K_n has the edge set $E(K_n) = \{\{i, j\} : 1 \leq i < j \leq n\}$. Below we displayed the complete graphs on 3 and 4 vertices. Obviously, K_n is closed with respect to any labeling of its vertices.

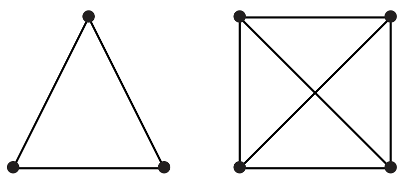


FIGURE 1.6. Complete graphs

The binomial edge ideal of K_n is the ideal $I_2(X)$ of all $2-$minors (maximal minors) of the $2 \times n$ matrix:

$$X = \begin{pmatrix} x_1 & \cdots & x_n \\ y_1 & \cdots & y_n \end{pmatrix}.$$

Hence we have

$$J_G = J_{K_n} = I_2(X) = (x_i y_j - x_j y_i : 1 \leq i < j \leq n) \subset S = K[x_1, \ldots, x_n, y_1, \ldots, y_n].$$

Let us list some properties of J_{K_n}.

(i) The complete graph is closed with respect to any labeling. Therefore, according to Theorem 1.4.2, the generators of J_{K_n} form a Gröbner basis.

(ii) Let $<$ be the lexicographic order on S induced by natural order of indeterminates. Then

$$\mathrm{in}_<(J_{K_n}) = (x_i y_j : 1 \leq i < j \leq n) = \bigcap_{i=1}^{n} (x_1, \ldots, x_{i-1}, y_{i+1}, \ldots, y_n).$$

(iii) We have $\dim\left(\frac{S}{J_{K_n}}\right) = \dim\left(\frac{S}{\text{in}_<(J_{K_n})}\right) = \dim(S) - \text{height}(\text{in}_<(J_{K_n}))$ $= 2n-(n-1) = n+1$.

(iv) $\frac{S}{J_{K_n}}$ is a domain [**13**, Theorem 6.35]. Hence, J_{K_n} is a prime ideal.

(v) $\text{in}_<(J_{K_n})$ has linear quotients.

(vi) Both $\text{in}_<(J_{K_n})$ and J_{K_n} have a linear resolution. Indeed, it is well known that a graded ideal generated in one degree which has linear quotients has also a linear resolution; see [**18**, Proposition 8.2.1]. This shows that $\text{in}_<(J_{K_n})$ has a linear resolution. For the second part we apply Corollary 1.2.6.

(vii) $\text{in}_<(J_{K_n})$ is Cohen-Macaulay.

(viii) J_{K_n} is Cohen-Macaulay, because according to Corollary 1.2.6, if $\text{in}_<(J_{K_n})$ is Cohen-Macaulay then J_{K_n} is also Cohen-Macaulay.

1.4.1.2. *The binomial edge ideal of the line graph.* Let $G = L_n$ be the line graph on the vertex set $[n]$ with $E(G) = \{(i,i+1) : 1 \leq i \leq n-1\}$. The binomial edge ideal J_G of L_n is $J_{L_n} = (f_{i,i+1} : 1 \leq i \leq n-1)$. Let us list some properties of J_{L_n}.

(i) The line graph L_n is closed with respect to the natural order of its vertices. Therefore, according to Theorem 1.4.2, the set of generators $\{f_{i,i+1} : 1 \leq i \leq n-1\}$ form a Gröbner basis for J_{L_n}. In fact it is possible to obtain the same conclusion without using Theorem 1.4.2. We know that the initial monomials of any two distinct generators of J_{L_n} are relatively prime. Then, we conclude that the generators of J_{L_n} form a Gröbner basis by Proposition 1.1.29.

(ii) $\text{in}_<(J_{L_n}) = (x_i y_{i+1} : 1 \leq i \leq n-1)$ is generated by a regular sequence of length $n-1$ of monomials of degree 2. It follows that the generators $f_{12}, f_{23}, \ldots, f_{n-1,n}$ of J_{L_n} form a regular sequence on S. This result is a consequence of the following lemma.

LEMMA 1.4.8. Let $I \subset S = K[x_1, \ldots x_n]$ be a graded ideal and $G = \{g_1, \ldots, g_m\}$ the reduced Gröbner basis of I with respect to $<$. If $\text{in}_<(g_1), \ldots, \text{in}_<(g_m)$ is a regular sequence in S, then $g_1, \ldots, g_m$ is a regular sequence in S.

(iii) Since J_{L_n} is a complete intersection, that is, J_{L_n} is generated by a regular sequence, it follows that J_{L_n} is Cohen-Macaulay. We have

$$\text{depth}\left(\frac{S}{J_{L_n}}\right) = \dim\left(\frac{S}{J_{L_n}}\right) = n+1.$$

1.4.2. Gröbner bases of binomial edge ideals. In general, for an arbitrary graph G, J_G has a Grobner basis whose initial monomials are squarefree.

In order to characterize in combinatorial terms the Grobner basis of J_G, we introduce the following definition.

DEFINITION 1.4.9. Let $i < j$ be two vertices of G. A path $i = i_0, i_1, \dots, i_{r-1}, i_r = j$ from i to j is called *admissible* if the following conditions are satisfied:

(i) $i_k \neq i_\ell$ for $k \neq \ell$;
(ii) for each $k = 1, \dots, r-1$, one has either $i_k < i$ or $i_k > j$;
(iii) for any proper subset $\{j_1, \dots, j_s\}$ of $\{i_1, \dots, i_{r-1}\}$, the sequence $i, j_1, \dots, j_s, j$ is not a path in G.

With a given admissible path π of G from i to j, we associate a monomial

$$u_\pi = (\prod_{i_k > j} x_{i_k})(\prod_{i_\ell < i} y_{i_\ell}).$$

Obviously, any edge of G is an admissible path. In this case, the associated monomial is just 1.

EXAMPLE 1.4.10. All the admissible paths other than the edges of C_5 with respect to the given labelling in Figure 1.7 are:

$$\pi_1 = 1,5,4; \quad \pi_2 = 2,1,5; \quad \pi_3 = 1,5,4,3; \quad \pi_4 = 2,1,5,4; \quad \pi_5 = 3,2,1,5.$$

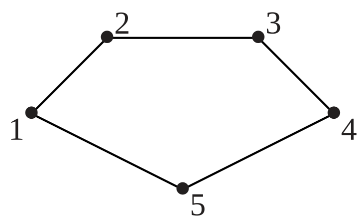


FIGURE 1.7

The associated monomials for these admissible paths are:

$$u_{\pi_1} = x_5; \quad u_{\pi_2} = y_1; \quad u_{\pi_3} = x_4x_5; \quad u_{\pi_4} = x_5y_1; \quad u_{\pi_5} = y_1y_2.$$

Note that, in a closed graph, the admissible paths are exactly the edges of G. Hence, if G is closed and connected then $\{i, i+1\}$ is an edge of G for any i.

THEOREM 1.4.11. [**19**] *The set of binomials*

$$\Gamma = \bigcup_{i<j} \{u_\pi f_{ij} : \pi \text{ is an admissible path from } i \text{ to } j\}$$

is the reduced Gröbner basis of J_G with respect to the lexicographic order on S induced by the natural order of indeterminates, $x_1 > \cdots > x_n > y_1 > \cdots > y_n$.

EXAMPLE 1.4.12. For the graph C_5 of figure 1.7 given in the previous example, the reduced Gröbner basis of J_{C_5} with respect to the lexicographic order is:

$$\{x_4x_5f_{13}, x_5y_1f_{24}, y_1y_2f_{35}, x_5f_{14}, y_1f_{25}, f_{12}, f_{15}, f_{23}, f_{34}, f_{45}\}.$$

As a consequence of Theorem 1.4.11, we see that all admissible paths of a graph G can be determined by computing the reduced Gröbner basis of J_G.

EXAMPLE 1.4.13. Let $G = K_{3,2}$ be a complete bipartite graph with 5 vertices given in Figure 1.8.

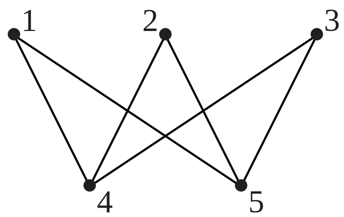


FIGURE 1.8

The admissible paths of $K_{3.2}$ other than the edges are:

$$\pi_1 = 1,4,2; \quad \pi_2 = 1,5,2; \quad \pi_3 = 1,4,3; \quad \pi_4 = 1,5,3; \quad \pi_5 = 2,4,3;$$
$$\pi_6 = 2,5,3; \quad \pi_7 = 4,1,5; \quad \pi_8 = 4,2,5; \quad \pi_9 = 4,3,5.$$

The reduced Gröbner basis of binomial edge ideal of the complete bipartite graph $G = K_{3,2}$ is given by $J_G = (f_{14}, f_{15}, f_{24}, f_{25}, f_{34}, f_{35}, x_4f_{12}, x_5f_{12}, x_4f_{13}, x_5f_{13}, x_4f_{23}, x_5f_{23}, y_1f_{45}, y_2f_{45}, y_3f_{45})$.

1.4.3. Primary decomposition of binomial edge ideals. J_G is a radical ideal [**19**, Corollary 2.2]. This is a consequence of Theorem 1.4.2. Since a radical ideal can be expessed as the intersection of its minimal prime ideals, we have

$$J_G = \bigcap_{P \in \mathrm{Min}(J_G)} P,$$

where $\mathrm{Min}(J_G)$ denotes the set of the minimal prime ideals of J_G.

We would like to characterize the minimal primes of J_G in terms of the combinatorics of G. We need to introduce the following notation.

Let G be a simple graph on $[n]$. For each subset $\mathscr{S} \subset [n]$ we define a prime ideal $P_{\mathscr{S}}$ in the following way. Let $G_1, \ldots, G_{c(\mathscr{S})}$ be the connected components of $G_{[n]\setminus\mathscr{S}}$, where $G_{[n]\setminus\mathscr{S}}$ is the induced subgraph of G on the vertex set $[n] \setminus \mathscr{S}$. For $1 \le i \le c(\mathscr{S})$, let $\widetilde{G}_i$ be the complete graph on the vertex set $V(G_i)$. We set

$$P_{\mathscr{S}}(G) = (\{x_i, y_i\}_{i\in\mathscr{S}}, J_{\tilde{G}_1}, \ldots, J_{\tilde{G}_{c(\mathscr{S})}}).$$

By using property (iv) of J_{K_n}, it follows that $P_{\mathscr{S}}(G)$ is a prime ideal since $J_{\tilde{G}_1}, \ldots, J_{\tilde{G}_{c(\mathscr{S})}}$ are prime binomial ideals whose generators belong to disjoint sets of variables.

We observe that, for any $\mathscr{S} \subset [n]$, $P_{\mathscr{S}}(G) \supset J_G$ and $\dim S/P_{\mathscr{S}}(G) = \sum_{i=1}^{c(\mathscr{S})} \dim(S_i/J_{\tilde{G}_i})$ where S_i is the polynomial ring in the variables x_j, y_j with $j \in V(G_i)$. Thus, we get

$$\dim S/P_{\mathscr{S}}(G) = \sum_{i=1}^{c(\mathscr{S})} (|V(G_i)|+1) = c(\mathscr{S}) + \sum_{i=1}^{c(\mathscr{S})} |V(G_i)| = c(\mathscr{S}) + n - |\mathscr{S}|.$$

THEOREM 1.4.14. [**19**] *Let G be a simple graph on the vertex set $[n]$. Then*

$$J_G = \bigcap_{\mathscr{S}\subset[n]} P_{\mathscr{S}}(G).$$

In particular, the minimal primes of J_G are among the prime ideals $P_{\mathscr{S}}(G)$, where $\mathscr{S} \subset [n]$.

The proof may be found in [**19**, Theorem 3.2].

COROLLARY 1.4.15. [**19**, Corollary 3.3] Let G be a simple graph on the vertex set $[n]$. Then

$$\dim S/J_G = \max\{n - |\mathscr{S}| + c(\mathscr{S}) : \mathscr{S} \subset [n]\}.$$

If we choose $\mathscr{S} = \emptyset$, then the number of connected components of G is $c = c(\emptyset)$. Since there is no variable in $P_\emptyset(G)$, it can be seen easily that $P_\emptyset(G)$ is not comparable to any other prime ideal $P_{\mathscr{S}}(G)$ where $\mathscr{S} \neq \emptyset$. Therefore $P_\emptyset(G)$ is a minimal prime of J_G. We calculate $\dim S/P_\emptyset(G) = n + c$ which is the maximum value of $n - |\mathscr{S}| + c(\mathscr{S})$, for instance, when J_G is Cohen-Macaulay. Because, in this case, J_G is unmixed which implies that all the minimal primes have the same dimension. In particular, if G is connected, then J_G is unmixed if and only if for every minimal prime $P_{\mathscr{S}}(G)$ of G, we have $n - |\mathscr{S}| + c(\mathscr{S}) = n + 1$, that is $c(\mathscr{S}) - |\mathscr{S}| = 1$.

Let us state a theorem that characterizes the sets $\mathscr{S}$ for which the prime ideal $P_{\mathscr{S}}(G)$ is minimal.

THEOREM 1.4.16. [**19**] *Let G be a connected graph on the vertex set $[n]$, and $\mathscr{S} \subset [n]$. Then $P_{\mathscr{S}}(G)$ is a minimal prime of J_G if and only if $\mathscr{S} = \emptyset$ or $\mathscr{S}$ is non-empty and for each $i \in \mathscr{S}$ one has $c(\mathscr{S} \setminus \{i\}) < c(\mathscr{S})$.*

For the proof, one may see [**19**, Corollary 3.9].

A set $\mathscr{S} \subset [n]$ satisfying the condition of the above theorem is called a *cut-point set* of G. The theorem simply says that if G is a connected graph, then $P_{\mathscr{S}}(G)$ is a minimal prime ideal of J_G if and only if each $i \in \mathscr{S}$ is a cut-point of the graph $G_{([n]\setminus\mathscr{S})\cup\{i\}}$.

EXAMPLE 1.4.17.

(1) The cut-point set for a complete graph G is the empty set.

(2) Let $G = L_n$ be the line graph on the vertex set $[n]$ with the natural labelling of the vertices. Then, a non-empty subset $\mathscr{S} \subset [n]$ is a cut-point set of G if and only if $\mathscr{S} = \{i_1, \ldots, i_r\}$ with $1 < i_1 < \cdots < i_r < n$ and $i_{s+1} - i_s > 1$ for all $1 \le s \le r-1$. For instance, let $G = L_5$ the line graph with 5 vertices. See Figure 1.9.

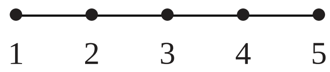


FIGURE 1.9

The the cut-point sets are $\emptyset, \{2\}, \{3\}, \{4\}, \{2,4\}$. Therefore, we may write

$$J_G = P_\emptyset(G) \cap P_{\{2\}}(G) \cap P_{\{3\}}(G) \cap P_{\{4\}}(G) \cap P_{\{2,4\}}(G)$$

as the intersection of the corresponding minimal primes. We have, for example, $P_{\{2,4\}}(G) = (x_2, y_2, x_4, y_4)$, $P_{\{2\}}(G) = (x_2, y_2, x_3y_4 - x_4y_3, x_3y_5 - x_5y_3, x_4y_5 - x_5y_4)$.

(3) Let G be a cycle of n vertices, $G = C_n$. A non-empty cut-point set $\mathscr{S} \subset [n]$ occurs when $|\mathscr{S}| > 1$ and no two elements $i, j \in \mathscr{S}$ belong to the same edge of C_n. J_G is not unmixed because $\dim P_\emptyset(G) = n+1$ and all the other minimal primes have dimension n. Here are, for instance, the minimal primes of $G = C_5$ given in Figure 1.7:

$$P_\emptyset(G),\ P_{\{1,3\}}(G),\ P_{\{1,4\}}(G),\ P_{\{2,4\}}(G),\ P_{\{2,5\}}(G),\ P_{\{3,5\}}(G).$$

(4) Let G be a graph on the vertex set $[7]$ shown in Figure 1.10.

The cut-point sets of G are $\emptyset, \{2\}, \{6\}, \{2,6\}, \{3,5\}, \{2,4,6\}$. Therefore we have

$$J_G = P_\emptyset(G) \cap P_{\{2\}}(G) \cap P_{\{6\}}(G) \cap P_{\{2,6\}}(G) \cap P_{\{3,5\}}(G) \cap P_{\{2,4,6\}}(G),$$

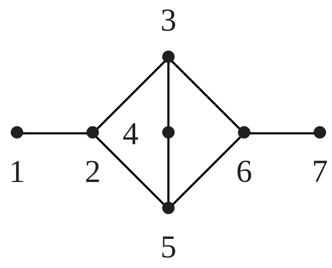


FIGURE 1.10

where, for example,

$$P_{\{2,6\}}(G) = (x_2, y_2, x_6, y_6, x_3y_4 - x_4y_3, x_3y_5 - x_5y_3, x_4y_5 - x_5y_4).$$

We calculate easily that $\dim S/J_G = 8$ for the maximum value of $n - |\mathscr{S}| + c(\mathscr{S})$ when $\mathscr{S} = \emptyset$.

1.4.4. The minimal graded free resolutions of some binomial edge ideals. We first consider the binomial edge ideal of the line graph. Let $G = L_n$ be the line graph on n vertices with $E(G) = \{\{i, i+1\} : 1 \leq i \leq n-1\}$.

As we have already seen, the generators $f = f_{12}, f_{23}, \ldots, f_{n-1,n}$ of J_{L_n} form a regular sequence in S. Therefore, the Koszul complex $K_\bullet(f)$ gives the minimal graded free resolution of S/J_{L_n} :

$$K_\bullet(f) : 0 \to K_{n-1}(f) \to \cdots \to K_j(f) \cdots \to K_1(f) \to K_0(f) = S \to S/J_{L_n} \to 0.$$

The S-module $K_j(f)$ is the j^{th} exterior power of the free S-module of rank $n-1$ of basis $e_1, \ldots, e_{n-1}$. Hence, $K_j(f)$ is also free over S of rank $\binom{n-1}{j}$ and basis

$$\{e_{i_1} \wedge \cdots \wedge e_{i_j} : 1 \leq i_1 < \cdots < i_j \leq n-1\}.$$

Since we would like to have all the maps in the above resolution of degree 0, we take $K_j(f) = S(-2j)^{\binom{n-1}{j}}$, for all j. Therefore,

$$\beta_{ij}(S/J_{L_n}) = \begin{cases} \binom{n-1}{i}, & j = 2i \\ 0, & \text{otherwise}. \end{cases}$$

On the other hand, the generators of $\mathrm{in}_<(J_{L_n})$ have the same property, namely they form a regular sequence in S. This implies that we have a similar resolution for $S/\mathrm{in}_<(J_{L_n})$. For the regularity we have

$$\operatorname{reg} S/J_{L_n} = \operatorname{reg}(S/\mathrm{in}_<(J_{L_n})) = n - 1.$$

We now consider the binomial edge ideal of the complete graph. Let $G = K_n$ be the complete graph on the vertex set $[n]$. As we have seen, in this case, J_{K_n} coincides with the ideal of all 2-minors of the matrix X whose rows are $x_1,\dots,x_n$ and $y_1,\dots,y_n$. The resolution of $I_2(X)$ is known, it is the Eagon-Northcott complex, but we would like to restrict to showing that $\beta_{ij}(S/\operatorname{in}_<(J_{K_n})) = \beta_{ij}(S/J_{K_n})$ also in this case, without using that complex.

Let us first observe that $\operatorname{in}_<(J_{K_n}) = (x_i y_j : 1 \le i < j \le n)$ has linear quotients if we order its generators in decreasing order with respect to the lexicographic order induced by $x_1 > \cdots > x_n > y_1 > \cdots > y_n$. Therefore, $S/\operatorname{in}_<(J_{K_n})$ and, consequently, S/J_{K_n} has a linear resolution, by Corollary 1.2.6. By Remark 1.2.4, the Betti numbers of S/J_{K_n} and of $S/\operatorname{in}_<(J_{K_n})$ are determined by their Hilbert series. But S/J_{K_n} and $S/\operatorname{in}_<(J_{K_n})$ have the same Hilbert series. Hence S/J_{K_n} and $S/\operatorname{in}_<(J_{K_n})$ have the same graded Betti numbers.

CHAPTER 2

Extremal Betti numbers of some classes of binomial edge ideals

In [**14**], the authors conjectured that the extremal Betti numbers of J_G and $\mathrm{in}_<(J_G)$ coincide for any graph G. Here, $<$ denotes the lexicographic order in $S = K[x_1,\dots,x_n,y_1,\dots,y_n]$ induced by the natural order of the variables $x_1 > \cdots > x_n > y_1 > \cdots > y_n$. In this section, we give a positive answer to this conjecture when the graph G is a complete bipartite graph or a cycle. To this aim, we use some results proved in [**29**] and [**32**] which completely characterize the resolution of the binomial edge ideal J_G when G is a cycle or a complete bipartite graph. In particular, in this case, it follows that J_G has a unique extremal Betti number. We recall all the known facts on the resolutions of binomial edge ideals of the complete bipartite graphs and cycles. We study the initial ideal of J_G when G is a bipartite graph or a cycle. We show that $\mathrm{proj\,dim}\,\mathrm{in}_<(J_G) = \mathrm{proj\,dim}\,J_G$ and $\mathrm{reg}\,\mathrm{in}_<(J_G) = \mathrm{reg}\,J_G$, and, therefore, $\mathrm{in}_<(J_G)$ has a unique extremal Betti number as well. Finally, we show that the extremal Betti number of $\mathrm{in}_<(J_G)$ is equal to that of J_G.

To our knowledge, this is the first attempt to prove the conjecture stated in [**14**] for extremal Betti numbers. In our study, we take advantage of the known results on the resolutions of binomial edge ideals of cycles and complete bipartite graphs and of the fact that their initial ideals have nice properties.

2.1. Binomial edge ideals of complete bipartite graphs

Let $G = K_{m,n}$ be the complete bipartite graph on the vertex set $\{1,..,m\} \cup \{m+1,..,m+n\}$ with $m \geq n \geq 1$ and let J_G be its binomial edge ideal. J_G

is generated by all the binomials $f_{ij} = x_i y_j - x_j y_i$ where $1 \le i \le m$ and $m+1 \le j \le m+n$. In [**29**, Theorem 5.3] it is shown that the Betti diagram of S/J_G has the form

	0	1	2	$\cdots$	p
0	1	0	0	$\cdots$	0
1	0	mn	0	$\cdots$	0
2	0	0	β_{24}	$\cdots$	$\beta_{p,p+2}$

where $p = \operatorname{proj}\dim S/J_G = \begin{cases} m, & \text{if } n = 1, \\ 2m+n-2, & \text{if } n > 1. \end{cases}$

In particular, from the above Betti diagram we may read that S/J_G has a unique extremal Betti number, namely $\beta_{p,p+2}$.

Moreover, in [**29**, Theorem 5.4] all the Betti numbers of S/J_G are computed. Since we are interested only in the extremal Betti number, we recall here its value as it was given in [**29**, Theorem 5.4], namely,

$$\beta_{p,p+2} = \begin{cases} m-1, & \text{if } p = m, \\ n-1, & \text{if } p = 2m+n-2. \end{cases}$$

One may easily see that the only admissible paths of the complete graph $G = K_{m,n}$ are the edges of G, the paths of the form $i, m+k, j$ with $1 \le i < j \le m$, $1 \le k \le n$, and $m+i, k, m+j$ with $1 \le i < j \le n$, $1 \le k \le m$. Therefore, we get the following consequence of Theorem 1.4.11.

COROLLARY 2.1.1. Let $G = K_{m,n}$ be the complete bipartite graph on the vertex set $V(G) = \{1,\dots,m\} \cup \{m+1,\dots,m+n\}$. Then

$$\operatorname{in}_<(J_G) = (\{x_i y_j\}_{\substack{1\le i\le m \\ m+1\le j\le m+n}}, \{x_i x_{m+k} y_j\}_{\substack{1\le i<j\le m \\ 1\le k\le n}}, \{x_{m+i} y_k y_{m+j}\}_{\substack{1\le i<j\le n \\ 1\le k\le m}}).$$

2.2. Binomial edge ideals of cycles

In this subsection, G denotes the n–cycle on the vertex set $[n]$ with edges $\{1,2\},\{2,3\},\dots,\{n-1,n\},\{1,n\}$.

In [**32**] it was shown that the Betti diagram of S/J_G has the form

	0	1	2	3	$\cdots$	n
0	1	0	0	0	$\cdots$	0
1	0	n	0	0	$\cdots$	0
2	0	0	β_{24}	0	$\cdots$	0
3	0	0	0	β_{36}	$\cdots$	0
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$
$n-2$	0	0	$\beta_{2,n}$	$\beta_{3,n+1}$	$\cdots$	$\beta_{n,2n-2}$

and all the Betti numbers were computed. One sees that we have a unique extremal Betti number and, by [**32**], we have $\beta_{n,2n-2} = \binom{n-1}{2} - 1$.

We now look at the initial ideal of J_G. It is obvious by Definition 1.4.9 and by the labeling of the vertices of G that the admissible paths are the edges of G and the paths of the form $i, i-1, \dots, 1, n, n-1, \dots, j+1, j$ with $2 \leq j-i \leq n-2$. Consequently, we get the following system of generators for the initial ideal of J_G.

COROLLARY 2.2.1. Let G be the n–cycle with the natural labeling of its vertices. Then

$$\mathrm{in}_<(J_G) = (x_1y_2, \dots, x_{n-1}y_n, x_1y_n, \{x_ix_{j+1}\cdots x_ny_1\cdots y_{i-1}y_j\}_{2\leq j-i\leq n-2}).$$

2.3. Extremal Betti numbers of complete bipartite graphs

Let $G = K_{m,n}$ be the complete bipartite graph on the vertex set $\{1,..,m\}\cup\{m+1,..,m+n\}$ with $m \geq n \geq 1$ and let J_G be its binomial edge ideal. The initial ideal $\mathrm{in}_<(J_G)$ has a nice property which is stated in the following proposition.

PROPOSITION 2.3.1. Let $G = K_{m,n}$ be the complete graph. Then $\mathrm{in}_<(J_G)$ has linear quotients.

THEOREM 2.3.2. *Let $G = K_{m,n}$ be the complete graph. Then*

$$\beta_{t,t+2}(\mathrm{in}_<(J_G)) = \sum_{\substack{1\leq i\leq m\\ m+1\leq j\leq m+n}} \binom{i+j-m-2}{t},$$

$$\beta_{t,t+3}(\mathrm{in}_<(J_G)) = \begin{cases} \sum_{\substack{1\leq i<j\leq m\\ 1\leq k\leq n}} \binom{n+k+j-3}{t}, & \text{if } n=1,\\ \sum_{\substack{1\leq i<j\leq m\\ 1\leq k\leq n}} \binom{n+k+j-3}{t} + \sum_{\substack{1\leq i<j\leq n\\ 1\leq k\leq m}} \binom{m+k+j-3}{t}, & \text{if } n>1. \end{cases}$$

In particular, by the above theorem, it follows the following corollary which shows that for $G = K_{m,n}$ the extremal Betti numbers of S/J_G and $S/\mathrm{in}_<(J_G)$ coincide.

COROLLARY 2.3.3. Let $G = K_{m,n}$ be the complete graph. Then we have

(a) $p = \mathrm{proj}\dim(S/\mathrm{in}_<(J_G)) = \mathrm{proj}\dim(\mathrm{in}_<(J_G)) + 1 = \begin{cases} m, & n=1,\\ 2m+n-2, & n>1.\end{cases}$

(b) $S/\mathrm{in}_<(J_G)$ has a unique extremal Betti number, namely

$$\beta_{p,p+2}(S/\mathrm{in}_<(J_G)) = \beta_{p-1,p+2}(\mathrm{in}_<(J_G)) = \begin{cases} m-1, & \text{if } n=1,\\ n-1, & \text{if } n>1.\end{cases}$$

2.4. Extremal Betti numbers of cycles

In this section, the graph G is an n–cycle. If $n=3$, then G is a complete graph, therefore the ideals J_G and $\mathrm{in}_<(J_G)$ have the same graded Betti numbers. Thus, in the sequel, we may consider $n \geq 4$.

As we have already seen in Corollary 2.2.1, $\mathrm{in}_<(J_G)$ is minimally generated by the initial monomials of the binomials corresponding to the edges of G and by $m = n(n-3)/2$ monomials of degree ≥ 3 which we denote by $v_1,\dots,v_m$ where we assume that if $i<j$, then either $\deg v_i < \deg v_j$ or $\deg v_i = \deg v_j$ and $v_i > v_j$. We observe that if $v_k = x_i x_{j+1}\cdots x_n y_1 \cdots y_{i-1} y_j$, we have $\deg v_k = n-j+i+1$. Hence, there are two monomials of degree 3, namely, $v_1 = x_1 x_n y_{n-1}$ and $v_2 = x_2 y_1 y_n$, three monomials of degree 4, namely, $v_3 = x_1 x_{n-1} x_n y_{n-2}, v_4 = x_2 x_n y_1 y_{n-1}, v_5 = x_3 y_1 y_2 y_n$, etc.

We introduce the following notation. We set $J = (x_1y_2, x_2y_3, \dots, x_{n-1}y_n)$, $I = J + (x_1 y_n)$, and, for $1 \leq k \leq m$, $I_k = I_{k-1} + (v_k)$, with $I_0 = I$. Therefore, $I_m = \mathrm{in}_<(J_G)$.

LEMMA 2.4.1. The ideals quotient $J : (x_1 y_n)$ and $I_{k-1} : (v_k)$, for $k \geq 1$, are minimally generated by regular sequences of monomials of length $n-1$.

REMARK 2.4.2. From the above proof we also note that if we build the monomial $v_k = x_i x_{j+1}\cdots x_n y_1 \cdots y_{i-1} y_j$, then the regular sequence of monomials which generates $I_{k-1} : (v_k)$ contains $j-i-2$ monomials of degree 2 and $n-j+i+1$ variables.

In the following lemma we compute the projective dimension and the regularity of S/I. This will be useful for the inductive study of the invariants of S/I_k.

LEMMA 2.4.3. We have $\mathrm{proj\,dim}\, S/I = n-1$ and $\mathrm{reg}\, S/I = n-2$.

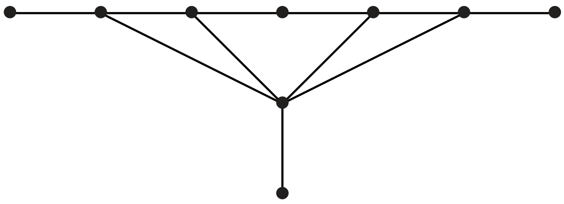
LEMMA 2.4.4. For $1 \leq k \leq m$, we have $\mathrm{proj\,dim}\, S/I_k \leq n$ and $\mathrm{reg}\, S/I_k \leq n-2$.

PROPOSITION 2.4.5. $\mathrm{proj\,dim}\, S/\mathrm{in}_<(J_G) = n$, $\mathrm{reg}\, S/\mathrm{in}_<(J_G) = n-2$.

THEOREM 2.4.6. *Let G be a cycle. Then $S/\mathrm{in}_<(J_G)$ and S/J_G have the same extremal Betti number, namely*

$$\beta_{n,2n-2}(S/J_G) = \beta_{n,2n-2}(S/\mathrm{in}_<(J_G)) = \binom{n-1}{2} - 1.$$

REMARK 2.4.7. There are examples of graphs whose binomial edge ideal have several extremal Betti numbers. For instance, the graph G displayed below has two extremal Betti numbers which are equal to the extremal Betti numbers of $\mathrm{in}_<(J_G)$.



CHAPTER 3

On the binomial edge ideals of block graphs

In this chapter we study homological properties of some classes of binomial edge ideals. Let G be a simple graph on the vertex set $[n]$ and let $S = K[x_1,\ldots,x_n,y_1,\ldots,y_n]$ be the polynomial ring in $2n$ variables over a field K.

We show that if G is a block graph, $\operatorname{depth}(S/J_G) = \operatorname{depth}(S/\operatorname{in}_<(J_G))$. Also we show a similar equality for regularity, namely

$$\operatorname{reg}(S/J_G) = \operatorname{reg}(S/\operatorname{in}_<(J_G)) = \ell \text{ if } G \text{ is a } C_\ell - \text{graph}.$$

C_ℓ-graphs constitute a subclass of the block graphs. In [**20**] it was shown that, for any connected graph G on the vertex set $[n]$, we have

$$\ell \leq \operatorname{reg}(S/J_G) \leq n-1,$$

where ℓ is the length of the longest induced path of G.

The main motivation of our work was to answer the following question. May we characterize the connected graphs G whose longest induced path has length ℓ and $\operatorname{reg}(S/J_G) = \ell$? We succeeded to answer this question for trees. We show that if T is a tree whose longest induced path has length ℓ, then $\operatorname{reg}(S/J_T) = \ell$ if and only if T is caterpillar. A *caterpillar tree* is a tree T with the property that it contains a path P such that any vertex of T is either a vertex of P or it is adjacent to a vertex of P.

In [**21**], the so-called weakly closed graphs were introduced. This is a class of graphs which includes closed graphs. In the same paper, it was shown that a tree is caterpillar if and only if it is a weakly closed graph. Having in mind our Theorem 3.2.1 and [**16**, Theorem 3.2] which states that $\operatorname{reg}(S/J_G) = \ell$ if G is a connected closed graph whose longest induced path has length ℓ, and by some computer experiments, we are tempted to formulate the following.

CONJECTURE 3.0.8. If G is a connected weakly closed graph whose longest induced path has length ℓ, then $\operatorname{reg}(S/J_G) = \ell$.

We first recall some basic definitions from graph theory. A vertex i of G whose deletion from the graph gives a graph with more connected components than G is called a *cut point* of G. A *chordal* graph is a graph without cycles of length greater than or equal to 4. A *clique* of a graph G is a complete subgraph of G. The cliques of a graph G form a simplicial complex, $\Delta(G)$, which is called the *clique complex* of G. Its facets are the maximal cliques of G. A graph G is a *block graph* if and only if it is chordal and every two maximal cliques have at most one vertex in common.

The clique complex $\Delta(G)$ of a chordal graph G has the property that there exists a *leaf order* on its facets. This means that the facets of $\Delta(G)$ may be ordered as $F_1,\dots,F_r$ such that, for every $i>1$, F_i is a leaf of the simplicial complex generated by $F_1,\dots,F_i$. A *leaf* F of a simplicial complex Δ is a facet of Δ with the property that there exists another facet of Δ, called a *branch* of F, say G, such that, for every facet $H \neq F$ of Δ, $H\cap F \subseteq G\cap F$.

3.1. Initial ideals of binomial edge ideals of block graphs

In this section we first show that for a block graph G on $[n]$ with c connected components $\operatorname{depth}(S/J_G) = \operatorname{depth}(S/\operatorname{in}_<(J_G)) = n+c$, where $<$ denotes the lexicographic order induced by $x_1 > \cdots > x_n > y_1 > \cdots > y_n$ in the ring $S = K[x_1,\dots,x_n,y_1,\dots,y_n]$.

We begin with the following lemma.

LEMMA 3.1.1. Let G be a graph on the vertex set $[n]$ and let $i \in [n]$. Then

$$\operatorname{in}_<(J_G,x_i,y_i) = (\operatorname{in}_<(J_G),x_i,y_i).$$

THEOREM 3.1.2. *Let G be a block graph and let c be the number of connected components of G. Then*

$$\operatorname{depth}(S/J_G) = \operatorname{depth}(S/\operatorname{in}_<(J_G)) = n+c.$$

Let G be a connected graph on the vertex set $[n]$ which consists of

(i) a sequence of maximal cliques $F_1,\dots,F_\ell$ with $\dim F_i \geq 1$ for all i such that $|F_i\cap F_{i+1}| = 1$ for $1 \leq i \leq \ell-1$ and $F_i \cap F_j = \emptyset$ for any $i<j$ such that $j \neq i+1$, together with

(ii) some additional edges of the form $F = \{j,k\}$ where j is an intersection point of two consecutive cliques F_i,F_{i+1} for some $1 \leq i \leq \ell-1$, and k is a vertex of degree 1.

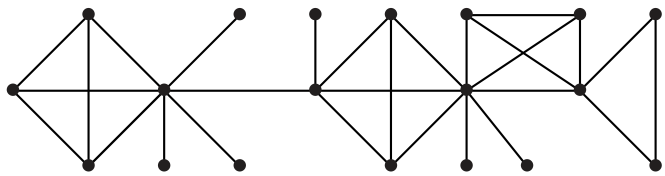


FIGURE 3.1. C_ℓ-graph

In other words, G is obtained from a graph H with $\Delta(H) = \langle F_1, \ldots, F_\ell \rangle$ whose binomial edge ideal is Cohen-Macaulay (see [**14**, Theorem 3.1]) by attaching edges in the intersection points of the facets of $\Delta(H)$. Therefore, G looks like the graph displayed in Figure 3.1.

Such a graph has, obviously, the property that its longest induced path has length equal to ℓ. If a connected graph G satisfies the above conditions (i) and (ii), we say that G is a $\mathscr{C}_\ell$-graph. In the case that $\dim F_i = 1$ for $1 \leq i \leq \ell$, then G is called a *caterpillar graph*.

We should also note that any $\mathscr{C}_\ell$–graph is chordal and has the property that any two distinct maximal cliques intersect in at most one vertex. So that any C_ℓ-graph is a connected block graph.

THEOREM 3.1.3. *Let G be a $\mathscr{C}_\ell$-graph on the vertex set $[n]$. Then*

$$\operatorname{reg}(S/J_G) = \operatorname{reg}(S/\operatorname{in}_<(J_G)) = \ell.$$

EXAMPLE 3.1.4. For the graph G of Figure 3.1 we get $\operatorname{reg}(S/J_G) = 5$.

3.2. Binomial edge ideals of caterpillar trees

Matsuda and Murai showed in [**20**] that, for any connected graph G on the vertex set $[n]$, we have

$$\ell \leq \operatorname{reg}(S/J_G) \leq n-1,$$

where ℓ denotes the length of the longest induced path of G, and conjectured that $\operatorname{reg}(S/J_G) = n-1$ if and only if T is a line graph. Several recent papers are concerned with this conjecture; see, for example, [**16**], [**25**], and [**27**]. One may ask as well to characterize connected graphs G whose longest induced path has length ℓ and $\operatorname{reg}(S/J_G) = \ell$. In this section, we answer this question for trees.

A caterpillar tree is a tree T with the property that it contains a path P such that any vertex of T is either a vertex of P or it is adjacent to a vertex of P. Clearly, any caterpillar tree is a $\mathscr{C}_\ell$-graph for some positive integer ℓ.

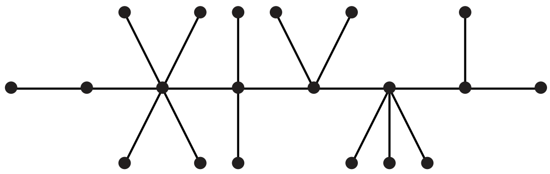


FIGURE 3.2. Caterpillar

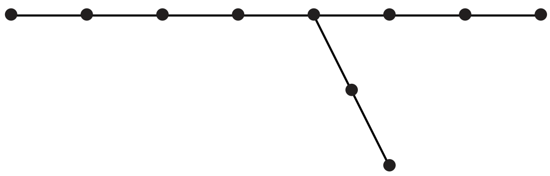


FIGURE 3.3. Induced graph H

Caterpillar trees were first studied by Harary and Schwenk [**17**]. These graphs have applications in chemistry and physics [**12**]. In Figure 3.2, an example of caterpillar tree is displayed. Note that any caterpillar tree is a narrow graph in the sense of Cox and Erskine [**5**]. Conversely, one may easily see that any narrow tree is a caterpillar tree. Moreover, as it was observed in [**21**], a tree is a caterpillar graph if and only if it is weakly closed in the sense of definition given in [**21**].

In the next theorem we characterize the trees T with $\operatorname{reg}(S/J_T) = \ell$ where ℓ is the length of the longest induced path of T.

THEOREM 3.2.1. *Let T be a tree on the vertex set $[n]$ whose longest induced path P has length ℓ. Then* $\operatorname{reg}(S/J_T) = \ell$ *if and only if T is caterpillar.*

CHAPTER 4

Binomial edge ideals and rational normal scrolls

Let K be a field and $S = K[x_1, \dots, x_{n+1}]$ the polynomial ring in $n+1$ variables over the field K. The 2-minors of the matrix

$$X = \begin{pmatrix} x_1 & \dots & x_{n-1} & x_n \\ x_2 & \dots & x_n & x_{n+1} \end{pmatrix}$$

generate the ideal I_X of the rational normal curve $\mathscr{X} \subset \mathbb{P}^n$. It is well-known that S/I_X is Cohen- Macaulay and has an S–linear resolution. We refer the reader to [**10**], [**4**], [**1**] for properties of the ideal of the rational normal scroll.

On the other hand, in the last few years, the so-called binomial edge ideals have been intensively studied. In analogy to the construction of classical binomial edge ideals, in this chapter we consider the following ideals in S. For a simple graph G on the vertex set $[n]$, let I_G be the ideal generated by the 2-minors $g_{ij} = x_i x_{j+1} - x_j x_{i+1}$ of X with $i < j$ and $\{i, j\} \in E(G)$. We call I_G the *binomial edge ideal of X*.

It is clear already from the beginning that unlike the case of classical binomial edge ideals, the ideal I_G strongly depends on the labeling of the graph G. For example, if G is the graph displayed in Figure 4.1, we get $\dim(S/I_G) = 3$ for the labeling given in Figure 4.2 (a) and $\dim(S/I_G) = 4$ for the labeling of G given in Figure 4.2 (b).

However, for some classes of graphs G which admit a natural labeling, we may associate with G a unique ideal I_G and study its properties. This is the case, for instance, for closed graphs. We recall from [**19**] that G is closed if it has a labeling with respect to which is closed. A graph G is called closed with respect to its given labeling if for all edges $\{i, j\}$ and

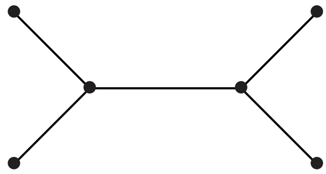
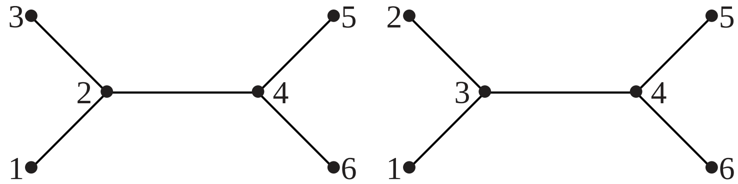


FIGURE 4.1



(a) (b)

FIGURE 4.2

$\{i,k\}$ with $j > i < k$ or $j < i > k$, one has $\{j,k\} \in E(G)$. A closed graph G is chordal and, therefore, by Dirac's Theorem, its clique complex $\Delta(G)$ is a quasi-forest. $\Delta(G)$ is a quasi-forest if the facets $F_1,\dots,F_r$ of $\Delta(G)$ have a leaf order. It was shown in [**14**] that if G is closed, then we may label the vertices of G such that the facets of $\Delta(G)$, say $F_1,\dots,F_r$, are intervals, $F_i = [a_i,b_i] \subset [n]$ and if we order $F_1,\dots,F_r$ such that $a_1 < a_2 < \cdots < a_r$, then this is a leaf order.

The rest of this chapter is structured as follows. In Section 4.1, we show that the generators of I_G form a Gröbner basis with respect to the reverse lexicographic order if and only if G is closed with the given labeling. As a consequence of this theorem, we derive that, for a closed graph G, the ideal I_G is Cohen-Macaulay of dimension $1+c$, where c is the number of connected components of G.

In Section 4.2, we study the properties of I_G for a closed graph G. We compute the minimal prime ideals of I_G in Theorem 4.2.2. By using this theorem, we characterize those connected closed graphs G for which I_G is a radical ideal (Proposition 4.2.3). In addition, we show, in Corollary 4.2.4, that I_G is a set-theoretic complete intersection if G is connected and closed. In the last part of Section 4.2, we give a sharp upper bound for the regularity of I_G (Theorem 4.2.7) and we show that I_G has a linear resolution if and only if G is a complete graph.

4.1. Gröbner bases

Let G be a graph on the vertex set $[n]$ and $I_G \subset S = K[x_1,\dots,x_n]$ its associated ideal. The main result of this section is the following.

THEOREM 4.1.1. *The generators of I_G form the reduced Gröbner basis of I_G with respect to the reverse lexicographic order induced by $x_1 > \cdots > x_n > x_{n+1}$ if and only if G is closed with respect to its given labeling.*

As in the case of classical binomial edge ideals associated with graphs, the ideal I_G where G is the line graph on n vertices has nice properties.

Let G be a line graph on $[n]$ with $E(G) = \{\{i, i+1\} : 1 \leq i \leq n-1\}$. Then I_G is minimally generated by $\{g_{i,i+1} = x_{i+1}^2 - x_i x_{i+2} : 1 \leq i \leq n-1\}$ and $\text{in}_{\text{rev}}(I_G) = (x_2^2, x_3^2, \ldots, x_n^2)$. As $x_2^2, x_3^2, \ldots, x_n^2$ is a regular sequence in S, it follows that the generators of I_G form a regular sequence as well. Consequently, the Koszul complex of the generators of I_G gives the minimal free resolution of S/I_G over S.

The following proposition shows that, for a closed graph G, the initial ideal of I_G with respect to the reverse lexicographic order has a simple structure.

PROPOSITION 4.1.2. Let G be a closed graph on $[n]$ with the clique complex $\Delta(G) = \langle F_1, \ldots, F_r \rangle$ where $F_i = [a_i, b_i]$ for $1 \leq i \leq r$, and $1 = a_1 < \cdots < a_r < b_r = n$. Then $\text{in}_{\text{rev}}(I_G)$ is a primary monomial ideal, hence it is Cohen-Macaulay.

COROLLARY 4.1.3. Let G be a closed graph. Then I_G is a Cohen-Macaulay ideal of $\dim(S/I_G) = 1 + c$ where c is the number of connected components of G.

4.2. Properties of the scroll binomial edge ideals of closed graphs

In this section we study several algebraic and homological properties of the ideal I_G where G is a closed graph on the vertex set $[n]$.

4.2.1. Associated primes. We recall that I_X denotes the binomial edge ideal associated with the complete graph K_n. It is well known that I_X is a prime ideal.

PROPOSITION 4.2.1. Let G be an arbitrary connected graph on the vertex set $[n]$. Then I_X is a minimal prime of I_G. If P is a minimal prime ideal of I_G which contains no variable, then $P = I_X$.

Now we restrict our study to ideals associated with connected closed graphs.

THEOREM 4.2.2. *Let $G \neq K_n$ be a connected closed graph on the vertex set $[n]$ and I_G its associated ideal. Then*

$$\text{Ass}(S/I_G) = \text{Min}(I_G) = \{I_X, (x_2, \ldots, x_n)\}.$$

As a consequence of the above theorem, we may characterize the radical ideals I_G.

PROPOSITION 4.2.3. Let G be a connected closed graph on the vertex set $[n]$. Then I_G is a radical ideal if and only if

$$G = K_n \text{ or } \Delta(G) = \langle [1, n-1], [2, n] \rangle.$$

Theorem 4.2.2 has the following nice consequence.

COROLLARY 4.2.4. Let G be a connected closed graph. Then I_G is a set-theoretic complete intersection.

4.2.2. Regularity. Let G be a closed graph on the vertex set $[n]$ and $I_G \subset S$ its associated ideal. The first question we may ask is under which conditions on the graph G the ideal I_G has a linear resolution. The next proposition answers this question. We first need the following known statement.

LEMMA 4.2.5. [**3**, Exercise 4.1.17 (c)] Let $R = K[x_1, \ldots, x_n]/I$ be a homogeneous Cohen-Macaulay ring. The ring R has an m-linear resolution if and only if $I_j = 0$ for $j < m$ and $\dim_K I_m = \binom{m+g-1}{m}$ where $g = \operatorname{height} I$.

PROPOSITION 4.2.6. Let G be a closed graph on $[n]$. Then the following are equivalent:

(a) G is a complete graph;
(b) I_G has a linear resolution;
(c) All powers of I_G have a linear resolution.

In the next theorem we give an upper bound for the regularity of I_G when G is a closed graph.

THEOREM 4.2.7. *Let G be a closed graph on the vertex set $[n]$. Then* $\operatorname{reg}(S/I_G) \leq r$ *where r is the number of maximal cliques of G.*

REMARK 4.2.8. The upper bound given in the above theorem is sharp. Indeed, let G be a closed graph with the maximal cliques $F_i = [a_i, a_{i+1}]$ where $1 = a_1 < a_2 < \cdots < a_r < a_{r+1} = n$. In this case, we have

$$\operatorname{in}_{\text{rev}}(I_G) = (x_2, \ldots, x_{a_2})^2 + (x_{a_2+1}, \ldots, x_{a_3})^2 + \cdots + (x_{a_r+1}, \ldots, x_n)^2.$$

Therefore,

$$S/(\operatorname{in}_{\text{rev}}(I_G), x_1, x_{n+1}) \cong (S_1/(x_2, \ldots, x_{a_2})^2) \otimes_K \cdots \otimes_K (S_r/(x_{a_r+1}, \ldots, x_n)^2),$$

where $S_i = K[x_{a_i+1}, \ldots, x_{a_{i+1}}]$ for all i, which implies that

$$H_{S/(\operatorname{in}_{\text{rev}}(I_G), x_1, x_{n+1})}(t) = \prod_{i=1}^{r} (1 + (a_{i+1} - a_i)t).$$

This shows that $\operatorname{reg}(S/I_G) = r$.

From Proposition 4.2.6 and Theorem 4.2.7, we derive the following consequence.

COROLLARY 4.2.9. Let G be a closed graph with two maximal cliques. Then $\mathrm{reg}(S/I_G) = 2$.

The following example shows that the inequality given in Theorem 4.2.7 may be also strict.

EXAMPLE 4.2.10. Let G be the closed graph on the vertex set $[6]$ with the maximal cliques $F_1 = [1,4], F_2 = [3,5], F_3 = [4,6]$. Then $\mathrm{reg}(S/I_G) = 2 < 3$.

Conclusions and further research

In summary, the main original results of this thesis are the following.

1. We showed that, if G is a complete bipartite graph or a cycle, then the associated binomial ideal J_G and its initial ideal $\mathrm{in}_<(J_G)$ with respect to the lexicographic order in $S = K[x_1, \ldots, x_n, y_1, \ldots, y_n]$ have the same extremal Betti numbers.
2. If G is a block graph on the vertex set $[n]$, then $\mathrm{depth}(S/J_G) = \mathrm{depth}(S/\mathrm{in}_<(J_G)) = n + c$ where c is the number of connected components of G.
3. If G is a C_ℓ-graph, then $\mathrm{reg}(S/J_G) = \mathrm{reg}(S/\mathrm{in}_<(J_G)) = \ell$. In particular, it follows that the binomial edge ideal of a C_ℓ–graph has minimal regularity.
4. We characterized the trees whose binomial edge ideals have minimal regularity.
5. We introduced binomial edge ideals of closed graphs associated with scrolls and we studied several algebraic and homological properties of them.

Binomial edge ideals have been intensively studied in the last 5 years. We intend to continue our research on this topic with a special focus on Matsuda and Murai conjecture [**20**] which states that, for a graph G on the vertex set $[n]$, we have $\mathrm{reg}(S/J_G) = n - 1$ if and only if G is the line graph. This conjecture was proved for block graphs in [**16**]. In particular, it follows that this conjecture holds for trees. Another interesting problem is to solve the conjecture on binomial edge ideals with minimal regularity which we proposed in [**8**].

Moreover, we would like to extend our research on binomial edge ideals associated with scrolls. For instance, one direction is to generalize our construction for a pair of graphs for a Hankel matrix of arbitrary type following ideas of the paper [**15**].

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